



Multivariate orthogonal polynomials: Quantum decomposition, deficiency rank and support of measure



Ameur Dhahri^a, Nobuaki Obata^b, Hyun Jae Yoo^{c,*}

^a *Dipartimento di Matematica, Politecnico di Milano, Piazza Leonardo da Vinci 32, I-20133 Milano, Italy*

^b *Graduate School of Information Sciences, Tohoku University, Sendai, 980-8579, Japan*

^c *Department of Applied Mathematics and Institute for Integrated Mathematical Sciences, Hankyong National University, 327 Jungang-ro, Anseong-si, Gyeonggi-do 17579, Republic of Korea*

ARTICLE INFO

Article history:

Received 17 April 2019

Available online 13 December 2019

Submitted by R.M. Aron

Keywords:

Multivariate orthogonal polynomials

Quantum decomposition

Favard theorem

Deficiency rank

Support of measure

ABSTRACT

In this paper we investigate the multivariate orthogonal polynomials based on the theory of interacting Fock spaces. Our framework is on the same stream line of the recent paper by Accardi, Barhoumi, and Dhahri [1]. The (classical) coordinate variables are decomposed into non-commuting (quantum) operators called creation, annihilation, and preservation operators, in the interacting Fock spaces. Getting the commutation relations, which follow from the commuting property of the coordinate variables between themselves, we can develop the reconstruction theory of the measure, namely the Favard's theorem. We then further develop some related problems including the marginal distributions and the rank theory of the Jacobi operators. We will see that the deficiency rank of the Jacobi operator implies that the underlying measure is supported on some algebraic surface and vice versa. We will provide with some examples.

© 2019 Elsevier Inc. All rights reserved.

1. Introduction

The aim of this paper is to develop the study of multivariate orthogonal polynomials within the formalism of interacting Fock spaces.

The univariate case has been studied in terms of one-mode interacting Fock spaces, where the Jacobi's three-term recurrence relation is transformed into annihilation, creation and preservation operators through the quantum decomposition of the multiplication operator by x . This aspect traces back to Accardi and Bożejko [2], and is now well understood with many applications, see e.g., [7] and references cited therein.

On the other hand, multivariate case has been also formulated within multi-mode interacting Fock spaces, where the coordinate variables are decomposed into a sum of creation, annihilation, and preservation operators in the interacting Fock space of the gradations of polynomials. In that case the Jacobi coefficients

* Corresponding author.

E-mail addresses: ameur.dhahri@polimi.it (A. Dhahri), obata@tohoku.ac.jp (N. Obata), yoohj@hknu.ac.kr (H.J. Yoo).

appearing in the three-term recurrence relation should be replaced with a pair of sequences of positive definite matrices and Hermitian ones [1,3,4]. Following the formulation established therein, we study multivariate version of Favard's theorem, and discuss the relation between the support of the probability measure and the Jacobi coefficients, where we use a newly introduced concept of deficiency rank.

Given a probability measure on \mathbb{R}^d with finite moments of all orders, we perform the Gram-Schmidt orthogonalization process and obtain the gradation spaces (spaces of polynomials of degree n for each $n \geq 0$ in the orthogonalization process). As is the case of univariate system, the coordinate variables x, y, \dots are decomposed into the three (non-commuting) operators between gradation spaces, which are called creation, annihilation, and preservation (or conservation) operators (called CAPs, hereafter). The commutativity of the (classical) variables themselves require some commutation relations for the CAPs. In view of this structure, we next consider the converse problem, namely the Favard's theory. Starting with an interacting Fock space provided with CAP operators which satisfy suitable conditions, we reconstruct a probability measure. For this we will use the spectral theory of mutually commuting operators after Xu [15,16]. We will see that the commutation relations are so strong that already for the simplest case of product measures, they must obey some rules to properly construct the measure. Moreover, we introduce a sequence $\{\rho_n = \text{rank } \Omega_n\}$ of the ranks of the Jacobi operators (matrices) and investigate the relation to the support of the measures. It would be interesting question to characterize probability measures in terms of the rank sequence $\{\rho_n\}$.

There are tremendous works on multivariate orthogonal polynomials from various aspects, see [6,9–11,14–19] and references cited therein. Our approach has an algebraic feature that enables us to use commutation relations of CAP operators and to discuss the supports of probability measures.

The organization of the paper is as follows. In Section 2, we shortly recall the univariate theory of orthogonal polynomials. In Section 3, we develop the interacting Fock spaces for the multivariate orthogonal polynomials by using CAPs. Section 4 deals with the reconstruction theory, which is a multivariate version of Favard's theorem. In Section 5, we introduce the form generators for the gradation spaces in the interacting Fock space. Section 6 is devoted to some examples. In Section 7 we deal with the marginals. In Section 8, we introduce the concept of deficiency rank of Jacobi operators and discuss the support of the measure.

2. Preliminary

In this section we briefly review the theory of univariate orthogonal polynomials. In this paper, by a measure on \mathbb{R}^d we mean a Borel measure without specifying the Borel σ -field anymore. Let μ be a probability measure on \mathbb{R} such that the moments of all orders exist. Let $\{p_n(x)\}$ be the monic orthogonal polynomials of μ obtained by Gram-Schmidt orthogonalization of $\{1, x, x^2, \dots\}$. Then there exist Jacobi sequences $\{\omega_n\}_{n \geq 1}$ and $\{\alpha_n\}_{n \geq 1}$ such that the three-term recurrence relation holds:

$$\begin{aligned} p_0(x) &= 1, \\ p_1(x) &= x - \alpha_1, \\ xp_n(x) &= p_{n+1}(x) + \alpha_{n+1}p_n(x) + \omega_n p_{n-1}(x), \quad n \geq 1. \end{aligned} \tag{2.1}$$

Here we notice that $\omega_n > 0$ for all $n \geq 1$ or there exists $m_0 \geq 1$ such that $\omega_n = 0$ for all $n \geq m_0$ and $\omega_n > 0$ for all $n < m_0$, and $\alpha_n \in \mathbb{R}$, $n \geq 1$ [7].

The Favard theorem says the converse: if there are Jacobi sequences $\{\omega_n\}_{n \geq 1}$ and $\{\alpha_n\}_{n \geq 1}$, then there is a probability measure on \mathbb{R} for which the polynomials $\{p_n(x)\}$, constructed via the three-term recurrence relation (2.1), are orthogonal.

Orthogonal polynomials can also be understood by using an interacting Fock space and CAP operators [2]. Let \mathcal{H} be the direct sum Hilbert space:

$$\mathcal{H} := \bigoplus_{n=0}^{\infty} \mathbb{C}^{\widehat{\otimes} n}.$$

For $n \geq 0$, let $\Phi_n := \mathbf{e}_1^{\otimes n}$, where $\mathbf{e}_1 := 1 \in \mathbb{C}$. Given a Jacobi sequence $(\{\omega_n\}, \{\alpha_n\})$, define linear operators A^+ , A^- , and A^0 on \mathcal{H} by

$$A^+ \Phi_n = \sqrt{\omega_{n+1}} \Phi_{n+1}, \quad n \geq 0, \tag{2.2}$$

$$A^0 \Phi_n = \alpha_{n+1} \Phi_n, \quad n \geq 0, \tag{2.3}$$

$$A^- \Phi_n = \sqrt{\omega_n} \Phi_{n-1}, \quad n \geq 1, \quad A^- \Phi_0 = 0. \tag{2.4}$$

The Jacobi coefficients, orthogonal polynomials, the interacting Fock space and the CAP operators have the following relations [7]:

Theorem 2.1. *Let $(\{\omega_n\}, \{\alpha_n\})$ be the Jacobi coefficients for a probability measure μ on \mathbb{R} having finite moments of any order and let $\{p_n(x)\}$ be the corresponding monic orthogonal polynomials, and let \mathcal{K} be the closure of the polynomial space in $L^2(\mathbb{R}, \mu)$. Let \mathcal{H} be the interacting Fock space with CAP operators in (2.2)–(2.4). Then the map*

$$U : \mathcal{H} \ni \Phi_n \mapsto (\omega_n \cdots \omega_1)^{-1/2} p_n \in \mathcal{K}, \quad n = 1, 2, \dots,$$

and defined by linear extension, is a unitary operator. It holds also that the multiplication operator by x defined on the polynomial space has a representation:

$$x = U(A^+ + A^0 + A^-)U^*,$$

which we call a quantum decomposition. Furthermore, the following relation for the moments holds:

$$\int_{-\infty}^{\infty} x^m d\mu(x) = \langle \Phi_0, (A^+ + A^0 + A^-)^m \Phi_0 \rangle_{\mathcal{H}}, \quad m = 0, 1, 2, \dots \tag{2.5}$$

Remark 2.2. The space \mathcal{K} is a proper subspace of $L^2(\mathbb{R}, \mu)$ unless the polynomial space is dense in $L^2(\mathbb{R}, \mu)$. A sufficient condition for the equality $\mathcal{K} = L^2(\mathbb{R}, \mu)$ is that μ is the solution to a determinate moment problem [5,7,20].

In this paper we extend the theory to the multivariate functions.

3. Interacting Fock spaces

In this section, given a probability measure on \mathbb{R}^d with finite moments of any order, we introduce an interacting Fock space and CAP operators. Then we represent the moments via vacuum expectation, which is definitely an extension of one-dimensional theory. All the basic ideas are already given in the reference [1], but here we deal with in a canonical setting and this will make the argument more clear.

Throughout this section, we fix a probability measure μ on \mathbb{R}^d such that the moments of μ of any order are finite. For a basic setting we follow [1]: we define “gradation” spaces. Let \mathcal{P}_n be the space of all polynomials of degree n . Here we emphasize that the spaces \mathcal{P}_n , $n \geq 0$, are understood as pre-Hilbert spaces equipped with a pre-scalar product $\langle \cdot, \cdot \rangle_{\mu}$, the L^2 -inner product. Thus any two polynomials f and g are equivalent, or regarded as the same vector if $\int |f - g|^2 d\mu = 0$, i.e., $f = g$ μ -a.e. Having this equivalence relation in mind we may think of \mathcal{P}_n ’s as Hilbertian subspaces of $L^2(\mathbb{R}^d, \mu)$.

We call $\mathcal{P}_n := \mathcal{P}_n \ominus \mathcal{P}_{n-1}$, $n = 0, 1, 2, \dots$, the n th gradation space ($\mathcal{P}_{-1} := \{0\}$). In other words, \mathcal{P}_n consists of polynomials of degree n subtracted by their orthogonal projections onto \mathcal{P}_{n-1} . We therefore get the following direct sum structure:

$$\mathcal{P}_n] = \bigoplus_{k=0}^n \mathcal{P}_k, \quad n = 0, 1, 2, \dots$$

From now on, the projection operators onto the spaces \mathcal{P}_n 's will be denoted by P_n 's and similarly by P_n 's for the projection operators onto \mathcal{P}_n 's. We let \mathcal{P} the space of all polynomials. Notice that $\mathcal{P} \subset \mathcal{K} := \bigoplus_{n=0}^{\infty} \mathcal{P}_n$. Below the constant unit function $1 \in \mathcal{P}$ is explicitly exposed whenever some operation is done on it.

For each $i = 1, \dots, d$, we understand the variable x_i also as a multiplication operator by x_i defined on \mathcal{P} . We notice here that when we consider x_i as an operator on $L^2(\mathbb{R}^d, \mu)$, it is an unbounded operator unless μ is compact supported. In that case we take \mathcal{P} as a common domain of x_i 's for $i = 1, \dots, d$, which is dense in $\mathcal{K} = \bigoplus_{n=0}^{\infty} \mathcal{P}_n$. We remark that \mathcal{K} may not be equal to $L^2(\mathbb{R}^d, \mu)$.

It is easy to check and has been shown in [1, Theorem 4.2] that

$$x_i P_n = P_{n+1} x_i P_n + P_n x_i P_n + P_{n-1} x_i P_n. \quad (3.1)$$

We can thus define creation, preservation, and annihilation operators on \mathcal{P} , denoted by a_i^+ , a_i^0 , a_i^- , $i = 1, \dots, d$, in that order, as follows.

$$a_i^+ |_{\mathcal{P}_n} := P_{n+1} x_i P_n, \quad (3.2)$$

$$a_i^0 |_{\mathcal{P}_n} := P_n x_i P_n, \quad (3.3)$$

$$a_i^- |_{\mathcal{P}_n} := P_{n-1} x_i P_n, \quad a_i^- |_{\mathcal{P}_0} := 0. \quad (3.4)$$

Therefore we have the following relation, called quantum decomposition.

$$x_i = a_i^+ + a_i^0 + a_i^-, \quad i = 1, \dots, d, \text{ on } \mathcal{P}. \quad (3.5)$$

As was shown in [1], we notice that $\{a_i^+ : i = 1, \dots, d\}$ is a set of mutually commuting operators. Moreover, $(a_i^+)^* |_{\mathcal{P}} = a_i^-$, and a_i^0 is a symmetric operator for each $i = 1, \dots, d$.

Now we transfer the story into the canonical interacting Fock space over \mathbb{C}^d . For each $n = 0, 1, 2, \dots$, we define the set of multi-indices:

$$\mathcal{I}^{(n)} \equiv \mathcal{I}_d^{(n)} := \{\mathbf{n} := (n_1, \dots, n_d) : n_i \geq 0, i = 1, \dots, d, |\mathbf{n}| := n_1 + \dots + n_d = n\}. \quad (3.6)$$

Let $\mathcal{H}_0 := \mathbb{C}$ and for each $n \geq 1$ let \mathcal{H}_n be the symmetric tensor product $(\mathbb{C}^d)^{\widehat{\otimes} n}$ equipped with a pre-scalar product $\langle \cdot, \cdot \rangle_n$ defined as follows: for $\mathbf{n} = (n_1, \dots, n_d)$, $\mathbf{m} = (m_1, \dots, m_d) \in \mathcal{I}^{(n)}$,

$$\langle \mathbf{e}_1^{\widehat{\otimes} n_1} \widehat{\otimes} \dots \widehat{\otimes} \mathbf{e}_d^{\widehat{\otimes} n_d}, \mathbf{e}_1^{\widehat{\otimes} m_1} \widehat{\otimes} \dots \widehat{\otimes} \mathbf{e}_d^{\widehat{\otimes} m_d} \rangle_n := \langle (a_1^+)^{n_1} \dots (a_d^+)^{n_d} 1, (a_1^+)^{m_1} \dots (a_d^+)^{m_d} 1 \rangle_{\mu}. \quad (3.7)$$

We identify $\mathcal{H}_0 \equiv \mathbb{C} \Phi_0$, where Φ_0 is any fixed unit vector (a symbol), called vacuum vector. Notice that $\{\mathbf{e}_1^{\widehat{\otimes} n_1} \widehat{\otimes} \dots \widehat{\otimes} \mathbf{e}_d^{\widehat{\otimes} n_d} : \mathbf{n} = (n_1, \dots, n_d) \in \mathcal{I}^{(n)}\}$ is an (not normalized) orthogonal basis for $(\mathbb{C}^d)^{\widehat{\otimes} n}$ with the canonical inner product, but it is not an orthogonal system for \mathcal{H}_n in general. For each $n \geq 0$, we define a linear operator $U_n : \mathcal{H}_n \rightarrow \mathcal{P}_n$ by

$$U_n(\mathbf{e}_1^{\widehat{\otimes} n_1} \widehat{\otimes} \dots \widehat{\otimes} \mathbf{e}_d^{\widehat{\otimes} n_d}) := (a_1^+)^{n_1} \dots (a_d^+)^{n_d} 1, \quad (3.8)$$

and by a linear extension. We easily check that U_n is a unitary isomorphism. We define an interacting Fock space:

$$\mathcal{H} := \bigoplus_{n=0}^{\infty} \mathcal{H}_n. \quad (3.9)$$

By defining $U := \bigoplus_{n=0}^{\infty} U_n$, the operator $U : \mathcal{H} \rightarrow \mathcal{K}$ becomes again a unitary isomorphism. We transfer the CAP operators into \mathcal{H} by

$$A_i^+ := U^* a_i^+ U, \quad A_i^0 := U^* a_i^0 U, \quad A_i^- := U^* a_i^- U, \quad i = 1, \dots, d. \tag{3.10}$$

Since the domain of CAP operators $\{a_i^+, a_i^0, a_i^- : i = 1, \dots, d\}$ as well as $x_i, i = 1, \dots, d$, is \mathcal{P} , the domain for the CAP operators $\{A_i^+, A_i^0, A_i^- : i = 1, \dots, d\}$ is $\mathcal{D} := U^{-1}\mathcal{P}$. Notice that any element $\sum_{n=0}^{\infty} \xi_n \in \mathcal{D}$ with $\xi_n \in \mathcal{H}_n$ has at most finitely many non-zero terms ξ_n .

By definition the creation operator A_i^+ has always a canonical form in the sense that for $\mathbf{n} = (n_1, \dots, n_d) \in \mathcal{I}^{(n)}$,

$$A_i^+ \Big|_{\mathcal{H}_n} (\mathbf{e}_1^{\widehat{\otimes} n_1} \widehat{\otimes} \dots \widehat{\otimes} \mathbf{e}_d^{\widehat{\otimes} n_d}) = \mathbf{e}_1^{\widehat{\otimes} n_1} \widehat{\otimes} \dots \widehat{\otimes} \mathbf{e}_i^{\widehat{\otimes} (n_i+1)} \widehat{\otimes} \dots \widehat{\otimes} \mathbf{e}_d^{\widehat{\otimes} n_d}. \tag{3.11}$$

The set of CAP operators $\{A_i^+, A_i^0, A_i^- : i = 1, \dots, d\}$ inherits the properties from the set of CAP operators $\{a_i^+, a_i^0, a_i^- : i = 1, \dots, d\}$. In particular we see that $\{A_i^+ : i = 1, \dots, d\}$ is a set of mutually commuting operators, $(A_i^+)^* \Big|_{\mathcal{D}} = A_i^-$, and A_i^0 is a symmetric operator for each $i = 1, \dots, d$. Let us define

$$X_i := A_i^+ + A_i^0 + A_i^-, \quad i = 1, \dots, d, \text{ on } \mathcal{D}. \tag{3.12}$$

Then

$$X_i = U^* x_i U, \quad i = 1, \dots, d, \text{ on } \mathcal{D}. \tag{3.13}$$

From the commutativity of $\{x_i : i = 1, \dots, d\}$, we see that $\{X_i : i = 1, \dots, d\}$ is a set of commuting operators. Moreover, the following commutation relations hold on the domain \mathcal{D} (see [1]): for all $j, k = 1, \dots, d$,

$$[A_j^+, A_k^+] = 0, \tag{3.14}$$

$$[A_j^+, A_k^0] + [A_j^0, A_k^+] = 0, \tag{3.15}$$

$$[A_j^+, A_k^-] + [A_j^0, A_k^0] + [A_j^-, A_k^+] = 0. \tag{3.16}$$

Taking adjoint, the relation (3.15) is equivalent to $[A_j^0, A_k^-] + [A_j^-, A_k^0] = 0$.

Now we have an interacting Fock space \mathcal{H} , the creation, annihilation, and preservation operators. It is then possible to compute the mixed moments of μ by the vacuum expectation.

Proposition 3.1. *For any $\mathbf{n} = (n_1, \dots, n_d) \in \mathcal{I}^{(n)}$, we have*

$$\begin{aligned} \int_{\mathbb{R}^d} x_1^{n_1} \dots x_d^{n_d} d\mu &= \langle \Phi_0, X_1^{n_1} \dots X_d^{n_d} \Phi_0 \rangle_0 \\ &= \langle \Phi_0, (A_1^+ + A_1^0 + A_1^-)^{n_1} \dots (A_d^+ + A_d^0 + A_d^-)^{n_d} \Phi_0 \rangle_0. \end{aligned} \tag{3.17}$$

Proof. We notice that

$$\int_{\mathbb{R}^d} x_1^{n_1} \dots x_d^{n_d} d\mu = \langle 1, x_1^{n_1} \dots x_d^{n_d} 1 \rangle_{\mu}.$$

The result now follows from the relation (3.13) and the fact that $U : \mathcal{H} \rightarrow \mathcal{K}$ is a unitary isomorphism. \square

Remark 3.2. The relation (3.17) is an extension of the univariate formula (2.5). By expansion, the r.h.s. of (3.17) is a linear combination of the terms:

$$\langle \Phi_0, A_1^{\epsilon_{1,1}} \cdots A_1^{\epsilon_{1,n_1}} \cdots A_d^{\epsilon_{d,1}} \cdots A_d^{\epsilon_{d,n_d}} \Phi_0 \rangle_0, \quad \epsilon_{i,k} \in \{+, 0, -\}.$$

It is clear that each term with $\epsilon_{1,1} + \cdots + \epsilon_{1,n_1} + \cdots + \epsilon_{d,1} + \cdots + \epsilon_{d,n_d} \neq 0$ is zero.

An example will be discussed in subsection 6.3.

4. Reconstruction theorem

In this section, we discuss the converse problem. That is, given an interacting Fock space over \mathbb{C}^d equipped with CAP operators we discuss how we can construct a probability measure on \mathbb{R}^d so that its interacting Fock space structure is the given one. From the discussion of the previous section, it is clear what kind of ingredients we must have at hand a priori. Suppose that we are given an interacting Fock space

$$\mathcal{H} := \bigoplus_{n=0}^{\infty} \mathcal{H}_n, \quad (4.1)$$

where \mathcal{H}_n is the vector space $(\mathbb{C}^d)^{\otimes n}$ equipped with a pre-scalar product $\langle \cdot, \cdot \rangle_n$. For $n \geq 0$, let P_n be the orthogonal projection onto n th component space, $\cdots \oplus \{0\} \oplus \mathcal{H}_n \oplus \{0\} \oplus \cdots$. We set $P_n := \sum_{k=0}^n P_k$. The creation operators $A_i^+ : \mathcal{H} \rightarrow \mathcal{H}$, $i = 1, \dots, d$, with a dense domain \mathcal{D} consisting of vectors with only finitely many non-zero components, are defined as in (3.11). We also define the operators A_i^- , $i = 1, \dots, d$, on the domain \mathcal{D} in such a way that $A_i^-|_{\mathcal{H}_{n+1}} : \mathcal{H}_{n+1} \rightarrow \mathcal{H}_n$ is the adjoint of $A_i^+|_{\mathcal{H}_n} : \mathcal{H}_n \rightarrow \mathcal{H}_{n+1}$. They are called the annihilation operators. By convention we let $A_i^- \Phi_0 := 0$. Suppose that for $i = 1, \dots, d$, we are also given preservation operators $A_i^0 : \mathcal{H} \rightarrow \mathcal{H}$, which are symmetric operators and $A_i^0|_{\mathcal{H}_n} : \mathcal{H}_n \rightarrow \mathcal{H}_n$. They may be all zero operators. On the dense subspace \mathcal{D} let us define the following operators

$$X_i := A_i^+ + A_i^0 + A_i^-, \quad i = 1, \dots, d. \quad (4.2)$$

We notice that X_i 's are symmetric operators on \mathcal{D} .

We will use the spectral theory for commuting self-adjoint operators. The following is sketched in [15, 16]. Recall that the self-adjoint operators T_1, \dots, T_d on a separable Hilbert space with spectral measures E_1, \dots, E_d , respectively, are said to be mutually commuting if their spectral measures commute, i.e.,

$$E_i(B)E_j(C) = E_j(C)E_i(B), \quad i, j = 1, \dots, d, \quad (4.3)$$

for any Borel sets B and C of \mathbb{R} . If T_1, \dots, T_d commute, then

$$E := E_1 \otimes \cdots \otimes E_d \quad (4.4)$$

is a spectral measure on \mathbb{R}^d , i.e., E is a projection valued measure such that

$$E(B_1 \times \cdots \times B_d) = E_1(B_1) \cdots E_d(B_d)$$

for any Borel sets $B_1, \dots, B_d \subset \mathbb{R}$. We call E the spectral measure of the commuting operators T_1, \dots, T_d .

When T_1, \dots, T_d are bounded, the condition (4.3) is equivalent to $T_i T_j = T_j T_i$, $i, j = 1, \dots, d$. However, if T_1, \dots, T_d are unbounded it is not the case in general, as the famous example by Nelson shows [13]. The following is a general spectral theorem which we will use, and it is summarized in [15]. Below, a vector Φ_0 is called a cyclic vector with respect to a commuting family of self-adjoint operators T_1, \dots, T_d on \mathcal{H} meaning that the space consisting of the vectors $P(T_1, \dots, T_n)\Phi_0$ is dense in \mathcal{H} , where P runs over the polynomials.

Theorem 4.1. Let \mathcal{H} be a separable Hilbert space and T_1, \dots, T_d be commuting family of self-adjoint operators on \mathcal{H} . If Φ_0 is a cyclic vector in \mathcal{H} with respect to T_1, \dots, T_d , then T_1, \dots, T_d are unitarily equivalent to the multiplication operators M_1, \dots, M_d , respectively,

$$(M_i f)(\mathbf{x}) = x_i f(\mathbf{x}), \quad 1 \leq i \leq d, \tag{4.5}$$

defined on $L^2(\mathbb{R}^d, \mu)$, where the measure μ is defined by $\mu(B) = \langle \Phi_0, E(B)\Phi_0 \rangle$. In particular if $\{T_i\}_{i=1}^d$ are bounded operators then μ is supported on a compact set $S \subset S_1 \times \dots \times S_d$ where for every $i = 1, \dots, d$, S_i is the spectrum of T_i .

In order to construct a probability measure starting from an interacting Fock space equipped with CAP operators, we will utilize the above theorem. Our main theorem in this section reads as follows.

Theorem 4.2. Suppose that there is a symmetric interacting Fock space over \mathbb{C}^d equipped with creation, annihilation, and preservation operators described in the beginning of this section. Suppose that the operators A_i^+ , A_i^0 , and A_i^- , $i = 1, \dots, d$, satisfy the following conditions:

- (i) $\|A_i^+|_{\mathcal{H}_n}(\xi_n)\|_{n+1} = 0$ and $\|A_i^0|_{\mathcal{H}_n}(\xi_n)\|_n = 0$ whenever $\|\xi_n\|_n = 0$;
- (ii) The commutation relations in (3.14)–(3.16) hold on \mathcal{D} ;
- (iii) The symmetric operators $\{X_i : i = 1, \dots, d\}$ defined in (4.2) are essentially self-adjoint. Moreover, the closures $\{\overline{X}_i : i = 1, \dots, d\}$ are mutually commuting.

Then there is a probability measure μ on \mathbb{R}^d such that its interacting Fock space constructed by the method in section 3 is the same as the one that we started with.

Remark 4.3. If the operators $\{X_i : i = 1, \dots, d\}$ are bounded, then the condition (iii) is automatically satisfied by Lemma 4.4 below and the fact that commuting in spectral measures is equivalent to commuting in operator themselves for bounded operators. In the case when they are not bounded a sufficient condition for (iii) will be given in Proposition 4.8.

For the proof of the theorem, we need the following observations.

Lemma 4.4. The operators X_i , $i = 1, \dots, d$, are mutually commuting on \mathcal{D} :

$$[X_j, X_k] = 0, \quad j, k = 1, \dots, d.$$

Proof. It follows directly from the commutation relations (3.14)–(3.16). \square

Lemma 4.5. Under the conditions (i) and (ii) of Theorem 4.2, Φ_0 is a cyclic vector w.r.t. $\{X_1, \dots, X_d\}$ on the symmetric interacting Fock space $\mathcal{H} = \bigoplus_{n=0}^{\infty} \mathcal{H}_n$. In particular for any $\mathbf{n} = (n_1, \dots, n_d) \in \mathcal{I}^{(n)}$, we have the equality

$$(A_1^+)^{n_1} \dots (A_d^+)^{n_d} \Phi_0 = P(X_1, \dots, X_d) \Phi_0, \tag{4.6}$$

where $P(x_1, \dots, x_d)$ is a polynomial of degree n .

Proof. Since any element of \mathcal{D} is a linear combination of the vectors $(A_1^+)^{n_1} \dots (A_d^+)^{n_d} \Phi_0$ it is enough to prove the relation (4.6). For the proof we use an inductive argument. For any $i = 1, \dots, d$, we see from (4.2) and the fact $A_i^- \Phi_0 = 0$ that

$$\begin{aligned} A_i^+ \Phi_0 &= (X_i - A_i^0) \Phi_0 \\ &= (X_i - a_i^0) \Phi_0, \end{aligned}$$

where $a_i^0 \in \mathbb{R}$ is the matrix component of the one-dimensional linear operator $A_i^0|_{\mathcal{H}_0}$. Suppose now that for any $\mathbf{k} = (k_1, \dots, k_d) \in \mathcal{I}^{(k)}$ for $k \leq n$, the claim holds:

$$(A_1^+)^{k_1} \dots (A_d^+)^{k_d} \Phi_0 = R(X_1, \dots, X_d) \Phi_0,$$

for some polynomial R of degree k . Now let $\mathbf{n} = (n_1, \dots, n_d) \in \mathcal{I}^{(n)}$. Then for any $i = 1, \dots, d$,

$$A_i^+ ((A_1^+)^{n_1} \dots (A_d^+)^{n_d}) \Phi_0 = (X_i - A_i^0 - A_i^-) ((A_1^+)^{n_1} \dots (A_d^+)^{n_d}) \Phi_0.$$

For the second term in the r.h.s., $A_i^0 ((A_1^+)^{n_1} \dots (A_d^+)^{n_d}) \Phi_0$, by definition of the operator A_i^0 , it is a linear combination of the vectors $(A_1^+)^{m_1} \dots (A_d^+)^{m_d} \Phi_0$, $\mathbf{m} = (m_1, \dots, m_d) \in \mathcal{I}^{(n)}$. Similarly by definition of A_i^- , the third term $A_i^- ((A_1^+)^{n_1} \dots (A_d^+)^{n_d}) \Phi_0$ is a linear combination of the vectors $(A_1^+)^{l_1} \dots (A_d^+)^{l_d} \Phi_0$, $\mathbf{l} = (l_1, \dots, l_d) \in \mathcal{I}^{(n-1)}$. By the induction hypothesis, the sum of those two terms is of the form $Q(X_1, \dots, X_d) \Phi_0$ for some polynomial of degree n . The first term is obviously of the form $X_i P(X_1, \dots, X_d) \Phi_0$, where we assumed $(A_1^+)^{n_1} \dots (A_d^+)^{n_d} \Phi_0 = P(X_1, \dots, X_d) \Phi_0$ for some polynomial P of degree n , which is also guaranteed by the induction hypothesis. The proof is now completed. \square

We are ready to prove Theorem 4.2.

Proof of Theorem 4.2. By abuse of the notations let us denote the closures of $\{X_i\}_{i=1}^d$ by the same symbols. Therefore X_i 's are mutually commuting self-adjoint operators. By Lemma 4.5, we see that Φ_0 is a cyclic vector in \mathcal{H} with respect to the commuting family of self-adjoint operators $\{X_i\}_{i=1}^d$. Thus by Theorem 4.1, each X_i is unitarily equivalent to a multiplication operator on $L^2(\mathbb{R}^d, \mu)$ for a probability measure μ on \mathbb{R}^d . In particular, we have

$$\langle \Phi_0, P(X_1, \dots, X_d) \Phi_0 \rangle = \int P(x_1, \dots, x_d) d\mu(\mathbf{x}), \quad (4.7)$$

for any polynomial $P(x_1, \dots, x_d)$. We notice that by Proposition 3.1 the interacting Fock space structure defined by this measure μ is unitarily equivalent to the one that we started with. The reconstruction has been established. \square

We emphasize here that for a practical application of Theorem 4.2, it is in general hard to show the essential self-adjointness of the operators $\{X_i\}_{i=1}^d$ unless we have some sufficient conditions for it. This is also observed in some other literature (see [15], for example). Below we give some sufficient conditions. A criterion for essential self-adjointness of semibounded operators developed by Jorgensen will be very much useful.

Theorem 4.6. ([8, Theorem 1]) *Let L be a semibounded and densely defined operator on a Hilbert space \mathcal{H} . Assume that there is an increasing sequence $\{P_n\}$ of self-adjoint projections in \mathcal{H} whose supremum coincides with the identity operator such that*

- (i) $\text{Ran}(P_n)$ is contained in $\text{Dom}(L)$ for all n ;
- (ii) There is a positive integer k such that the range of LP_n is contained in that of P_{n+k} for all n ;

(iii) $\|(I - P_n)LP_n\| \leq a_n$ for some sequence $\{a_n\}$ of positive numbers satisfying

$$\sum_{n=1}^{\infty} a_n^{-1/2} = \infty.$$

Then, the restriction of L to $\cup_n \text{Ran}(P_n)$ is essentially self-adjoint.

We also need the following lemma which is due to Nelson [12] (cf. [15]).

Lemma 4.7. *Let T and S be symmetric operators on a Hilbert space \mathcal{H} and let \mathcal{D} be a dense subspace of \mathcal{H} such that \mathcal{D} is contained in the domain of T^2 , S^2 , TS , and ST , and $TS\psi = ST\psi$ for all $\psi \in \mathcal{D}$. If the restriction of $S^2 + T^2$ to \mathcal{D} is essentially self-adjoint then T and S are essentially self-adjoint and \overline{T} and \overline{S} commute, where \overline{T} stands for the closure of T .*

Proposition 4.8. *Suppose that there is a sequence $\{a_n\}$ of positive numbers satisfying $\sum_{n=0}^{\infty} a_n^{-1/2} = \infty$ and such that*

$$\|(I - P_n)X_i^2P_n\| \leq a_n, \quad i = 1, \dots, d. \tag{4.8}$$

Then $\{X_i : i = 1, \dots, d\}$ are essentially self-adjoint and the closures $\{\overline{X}_i : i = 1, \dots, d\}$ are mutually commuting.

For a proof, we need the following.

Lemma 4.9. *Under the hypothesis of Proposition 4.8 the operators X_i^2 and their sums $\{X_i^2 + X_j^2\}$, $i, j = 1, \dots, d$, are essentially self-adjoint on the domain \mathcal{D} .*

Proof. For each $i = 1, \dots, d$, since X_i is symmetric X_i^2 is positive definite on \mathcal{D} . Thus it is bounded from below. Notice that $\cup_{n \geq 0} \text{Ran}(P_n) = \mathcal{D}$ and $\text{Ran}X_i^2P_n \subset \text{Ran}P_{n+2}$. The result now follows from Theorem 4.6. The same argument applies also to the sums $X_i^2 + X_j^2$. \square

Proof of Proposition 4.8. The proof follows from Lemma 4.7 and Lemma 4.9. \square

The following should be practically useful.

Proposition 4.10. *For each $n \geq 0$ and $i = 1, \dots, d$, define $a_{n,i} := \|A_i^+|_{\mathcal{H}_n}\|$ and $b_{n,i} := \|A_i^0|_{\mathcal{H}_n}\|$. Suppose that*

$$\sum_{n=0}^{\infty} \frac{1}{a_{n,i} + b_{n,i} + a_{n+1,i} + b_{n+1,i}} = \infty, \quad 1 \leq i \leq d.$$

Then, the inequality (4.8) holds with $a_n = (a_{n,i} + b_{n,i} + a_{n+1,i} + b_{n+1,i})^2$, and hence the conclusion of Proposition 4.8 holds.

Proof. From the decomposition $X_i = A_i^+ + A_i^0 + A_i^-$, we have

$$\begin{aligned} (I - P_n)X_i^2P_n &= (I - P_n)(A_i^+ + A_i^0 + A_i^-)^2P_n \\ &= (I - P_n)((A_i^+)^2 + A_i^+A_i^0 + A_i^0A_i^+P_n), \end{aligned}$$

where we have used the property of the CAP operators and the gradation of the projections. For example, $\text{Ran}(A_i^0 A_i^0 P_n]$ belongs to $P_n]$ and it is orthogonal to $I - P_n]$. By the assumption, we see that

$$\|(I - P_n]X_i^2 P_n]\| \leq a_{n,i}a_{n+1,i} + a_{n,i}b_{n,i} + a_{n,i}b_{n+1,i} \leq (a_{n,i} + b_{n,i} + a_{n+1,i} + b_{n+1,i})^2.$$

The proof is completed. \square

Remark 4.11. We notice that the condition in Proposition 4.10 is similar but slightly stronger than the one given in [15, Theorem 2].

4.1. Example: univariate Favard’s theory and product measures

In this subsection we discuss one-dimensional theory and product measures.

Univariate Favard’s theorem. First we consider the one-mode interacting Fock space. Let $(\{\omega_n\}, \{\alpha_n\})$ be a Jacobi sequence as in section 2. Let $\mathcal{H}_0 := \mathbb{C}\mathbf{1}$ and for $n \geq 1$, let $\mathcal{H}_n := \mathbb{C}^{\widehat{\otimes} n}$ equipped with the inner product defined by

$$\langle \mathbf{e}^{\widehat{\otimes} n}, \mathbf{e}^{\widehat{\otimes} n} \rangle_n := \prod_{k=1}^n \omega_k \tag{4.9}$$

and by linear extension. We define A^+ , A^0 , and A^- as follows:

$$\begin{aligned} A^+|_{\mathcal{H}_n} &: \mathbf{e}^{\widehat{\otimes} n} \mapsto \mathbf{e}^{\widehat{\otimes}(n+1)} \in \mathcal{H}_{n+1}, \\ A^0|_{\mathcal{H}_n} &: \mathbf{e}^{\widehat{\otimes} n} \mapsto \alpha_{n+1} \mathbf{e}^{\widehat{\otimes} n} \in \mathcal{H}_n, \\ A^-|_{\mathcal{H}_n} &: \mathbf{e}^{\widehat{\otimes} n} \mapsto \omega_n \mathbf{e}^{\widehat{\otimes}(n-1)} \in \mathcal{H}_{n-1}, \quad n \geq 1, \\ A^-|_{\mathcal{H}_0} &:= 0, \end{aligned}$$

and by linear extension. It is promptly checked that $A^- = (A^+)^*$ and the properties (i) and (ii) in the statement of Theorem 4.2 are satisfied. Thus if the operator norms of $A^+|_{\mathcal{H}_n}$ and $A^0|_{\mathcal{H}_n}$ are moderate to satisfy the condition (4.8), which amounts to saying that the Jacobi sequences do not increase too fast, then by Theorem 4.2 there is a probability measure μ on \mathbb{R} such that the sequences $\{\omega_n\}$ and $\{\alpha_n\}$ are the Jacobi sequences corresponding to the measure μ .

Product measures. Let μ_1 and μ_2 be two probability measures on \mathbb{R} with Jacobi sequences $(\{\omega_n\}, \{\alpha_n\})$ and $(\{\eta_n\}, \{\beta_n\})$, respectively. Let \mathcal{H}_n be the vector space $(\mathbb{C}^2)^{\widehat{\otimes} n}$ equipped with an inner product defined as follows: for $\mathbf{n} = (n_1, n_2)$, $\mathbf{m} = (m_1, m_2) \in \mathcal{I}_2^{(n)}$,

$$\langle \mathbf{e}_1^{\widehat{\otimes} n_1} \widehat{\otimes} \mathbf{e}_2^{\widehat{\otimes} n_2}, \mathbf{e}_1^{\widehat{\otimes} m_1} \widehat{\otimes} \mathbf{e}_2^{\widehat{\otimes} m_2} \rangle_n := \delta_{n_1, m_1} \delta_{n_2, m_2} \prod_{k=1}^{n_1} \omega_k \prod_{l=1}^{n_2} \eta_l, \tag{4.10}$$

and by linear extension. Define the creation, preservation, and annihilation operators as follows: for $\mathbf{n} = (n_1, n_2) \in \mathcal{I}_2^{(n)}$,

$$\begin{aligned} A_1^+|_{\mathcal{H}_n} &: \mathbf{e}_1^{\widehat{\otimes} n_1} \widehat{\otimes} \mathbf{e}_2^{\widehat{\otimes} n_2} \mapsto \mathbf{e}_1^{\widehat{\otimes}(n_1+1)} \widehat{\otimes} \mathbf{e}_2^{\widehat{\otimes} n_2} \in \mathcal{H}_{n+1}, \\ A_2^+|_{\mathcal{H}_n} &: \mathbf{e}_1^{\widehat{\otimes} n_1} \widehat{\otimes} \mathbf{e}_2^{\widehat{\otimes} n_2} \mapsto \mathbf{e}_1^{\widehat{\otimes} n_1} \widehat{\otimes} \mathbf{e}_2^{\widehat{\otimes}(n_2+1)} \in \mathcal{H}_{n+1}, \\ A_1^0|_{\mathcal{H}_n} &: \mathbf{e}_1^{\widehat{\otimes} n_1} \widehat{\otimes} \mathbf{e}_2^{\widehat{\otimes} n_2} \mapsto \alpha_{n_1+1} \mathbf{e}_1^{\widehat{\otimes} n_1} \widehat{\otimes} \mathbf{e}_2^{\widehat{\otimes} n_2} \in \mathcal{H}_n, \end{aligned}$$

$$\begin{aligned}
 A_2^0|_{\mathcal{H}_n} &: \mathbf{e}_1^{\widehat{\otimes} n_1} \widehat{\otimes} \mathbf{e}_2^{\widehat{\otimes} n_2} \mapsto \beta_{n_2+1} \mathbf{e}_1^{\widehat{\otimes} n_1} \widehat{\otimes} \mathbf{e}_2^{\widehat{\otimes} n_2} \in \mathcal{H}_n, \\
 A_1^-|_{\mathcal{H}_n} &: \mathbf{e}_1^{\widehat{\otimes} n_1} \widehat{\otimes} \mathbf{e}_2^{\widehat{\otimes} n_2} \mapsto \omega_{n_1} \mathbf{e}_1^{\widehat{\otimes} (n_1-1)} \widehat{\otimes} \mathbf{e}_2^{\widehat{\otimes} n_2} \in \mathcal{H}_{n-1}, \quad n \geq 1, \\
 A_2^-|_{\mathcal{H}_n} &: \mathbf{e}_1^{\widehat{\otimes} n_1} \widehat{\otimes} \mathbf{e}_2^{\widehat{\otimes} n_2} \mapsto \eta_{n_2} \mathbf{e}_1^{\widehat{\otimes} n_1} \widehat{\otimes} \mathbf{e}_2^{\widehat{\otimes} (n_2-1)} \in \mathcal{H}_{n-1}, \quad n \geq 1, \\
 A_1^-|_{\mathcal{H}_0} &:= 0, \quad A_2^-|_{\mathcal{H}_0} := 0,
 \end{aligned}$$

and by linear extension. It is easy to check that $A_i^- = (A_i^+)^*$, $i = 1, 2$, and the properties (i) and (ii) of Theorem 4.2 are satisfied. Thus by Theorem 4.2, if the Jacobi sequences are moderate to satisfy (4.8) there is a probability measure μ on \mathbb{R}^2 whose interacting Fock space structure reproduces the one given above. We can obviously extend the argument to any d -dimensional product measures. We will consider other example in subsection 6.3.

5. CAP operators and the form generator

Recall that the pre-Hilbert space \mathcal{H}_n in the interacting Fock space related to a probability measure on \mathbb{R}^d is the vector space $(\mathbb{C}^d)^{\widehat{\otimes} n}$ equipped with a pre-scalar product $\langle \cdot, \cdot \rangle_n$. As a reference, we also regard $(\mathbb{C}^d)^{\widehat{\otimes} n}$ as a Hilbert space equipped with the canonical inner product, which we denote by $(\cdot, \cdot)_0$. We let $\mathcal{H}_{n,0} := ((\mathbb{C}^d)^{\widehat{\otimes} n}, (\cdot, \cdot)_0)$. Since $\langle \cdot, \cdot \rangle_n$ defines a positive definite quadratic form on $\mathcal{H}_{n,0}$, there is a positive definite operator $\Omega_n : \mathcal{H}_{n,0} \rightarrow \mathcal{H}_{n,0}$ such that

$$\langle \cdot, \cdot \rangle_n = (\cdot, \Omega_n \cdot)_0. \tag{5.1}$$

From the theory developed before we easily get a matrix representation of Ω_n by using the creation operators: for $\mathbf{n} = (n_1, \dots, n_d)$, $\mathbf{m} = (m_1, \dots, m_d) \in \mathcal{I}^{(n)}$,

$$\begin{aligned}
 &(\mathbf{e}_1^{\widehat{\otimes} n_1} \widehat{\otimes} \dots \widehat{\otimes} \mathbf{e}_d^{\widehat{\otimes} n_d}, \Omega_n \mathbf{e}_1^{\widehat{\otimes} m_1} \widehat{\otimes} \dots \widehat{\otimes} \mathbf{e}_d^{\widehat{\otimes} m_d})_0 \\
 &= \langle \mathbf{e}_1^{\widehat{\otimes} n_1} \widehat{\otimes} \dots \widehat{\otimes} \mathbf{e}_d^{\widehat{\otimes} n_d}, \mathbf{e}_1^{\widehat{\otimes} m_1} \widehat{\otimes} \dots \widehat{\otimes} \mathbf{e}_d^{\widehat{\otimes} m_d} \rangle_n \\
 &= \langle (A_1^+)^{n_1} \dots (A_d^+)^{n_d} \Phi_0, (A_1^+)^{m_1} \dots (A_d^+)^{m_d} \Phi_0 \rangle_n.
 \end{aligned} \tag{5.2}$$

By this we see that given a probability measure on \mathbb{R}^d , we get an interacting Fock space $\mathcal{H} = \bigoplus_{n=0}^\infty \mathcal{H}_n$, a sequence of positive definite operators $\Omega_n : \mathcal{H}_{n,0} \rightarrow \mathcal{H}_{n,0}$, and sequences of Hermitian operators $B_{i|n} : \mathcal{H}_n \rightarrow \mathcal{H}_n$, $i = 1, \dots, d$, which are in fact defined by $B_{i|n} := A_i^0|_{\mathcal{H}_n}$. It is worth mentioning that the creation and annihilation operators play a role in the definition of Ω_n implicitly and moreover, the operators satisfy the conditions (i), (ii), and (iii) in the statement of Theorem 4.2.

Next let us consider the converse problem. So, suppose that there is a sequence of positive definite operators $\Omega_n : \mathcal{H}_{n,0} \rightarrow \mathcal{H}_{n,0}$, and for each $i = 1, \dots, d$, suppose that there is a sequence of Hermitian operators $B_{i|n} : \mathcal{H}_{n,0} \rightarrow \mathcal{H}_{n,0}$. Our purpose is to see under what conditions one can construct a probability measure on \mathbb{R}^d . We proceed in the following steps.

- (i) Interacting Fock space. We can define an interacting Fock space as follows. Define a pre-scalar product $\langle \cdot, \cdot \rangle_n := (\cdot, \Omega_n \cdot)_0$ and let $\mathcal{H}_n := ((\mathbb{C}^d)^{\widehat{\otimes} n}, \langle \cdot, \cdot \rangle_n)$. The interacting Fock space is denoted by $\mathcal{H} := \bigoplus_{n=0}^\infty \mathcal{H}_n$.
- (ii) Creation and annihilation operators. As usual we define $A_i^+ : \mathcal{H} \rightarrow \mathcal{H}$, $i = 1, \dots, d$, by

$$A_i^+|_{\mathcal{H}_n} : \mathbf{e}_1^{\widehat{\otimes} n_1} \widehat{\otimes} \dots \widehat{\otimes} \mathbf{e}_d^{\widehat{\otimes} n_d} \mapsto \mathbf{e}_1^{\widehat{\otimes} n_1} \widehat{\otimes} \dots \widehat{\otimes} \mathbf{e}_i^{\widehat{\otimes} (n_i+1)} \widehat{\otimes} \dots \widehat{\otimes} \mathbf{e}_d^{\widehat{\otimes} n_d},$$

and by a linear extension. We let A_i^- be the adjoint of A_i^+ for $i = 1, \dots, d$, by defining $A_i^-|_{\mathcal{H}_n} := (A_i^+|_{\mathcal{H}_{n-1}})^*$ and $A_i^-|_{\mathcal{H}_0} := 0$.

(iii) Preservation operators. We let $A_i^0 := \bigoplus_{n=0}^\infty B_{i|n}$, $i = 1, \dots, d$.

We are now ready to state another reconstruction theorem.

Theorem 5.1. *Given a probability measure on \mathbb{R}^d , there is a sequence of positive definite operators $\{\Omega_n\}$ satisfying (5.1)–(5.2), and for each $i = 1, \dots, d$, there is a sequence of Hermitian operators $B_{i|n} : \mathcal{H}_n \rightarrow \mathcal{H}_n$. On the other hand, suppose that we are given a sequence of positive definite operators $\Omega_n : \mathcal{H}_{n,0} \rightarrow \mathcal{H}_{n,0}$, and for each $i = 1, \dots, d$, a sequence of Hermitian operators $B_{i|n} : \mathcal{H}_{n,0} \rightarrow \mathcal{H}_{n,0}$ so that we could construct an interacting Fock space, creation, annihilation, and preservation operators via the process (i)–(iii) described above. Suppose that the system of operators $\{A_i^+, A_i^0, A_i^- : i = 1, \dots, d\}$ satisfy the conditions (i), (ii), and (iii) in the statement of Theorem 4.2. Then there is a probability measure μ on \mathbb{R}^d such that the operators $\{\Omega_n\}$ and $\{B_{i|n}\}$ are reconstructed from the measure μ .*

Proof. By taking $B_{i|n} := A_i^0|_{\mathcal{H}_n}$, the forward direction was already observed above. The converse follows from Theorem 4.2. \square

Example 5.2. In this example let us consider the simplest example for two dimensional space, namely we consider the case where Ω_n 's are diagonal and $B_{i|n} \equiv 0$ for all $n \geq 0$ and $i = 1, 2$. For $\mathbf{n} = (n_1, n_2)$, $\mathbf{m} = (m_1, m_2) \in \mathcal{I}_2^{(n)}$, let

$$(\mathbf{e}_1^{\widehat{\otimes} m_1} \widehat{\otimes} \mathbf{e}_2^{\widehat{\otimes} m_2}, \Omega_n \mathbf{e}_1^{\widehat{\otimes} n_1} \widehat{\otimes} \mathbf{e}_2^{\widehat{\otimes} n_2})_0 := \delta_{m_1, n_1} \delta_{m_2, n_2} d_{m_1}^{(n)}, \tag{5.3}$$

where $d_k^{(n)}$'s, $k = 0, \dots, n$, are positive diagonal components of Ω_n . Recall by step (i) mentioned in this section that the inner product $\langle \cdot, \cdot \rangle_n$ on $(\mathbb{C}^2)^{\widehat{\otimes} n}$ is defined for $\mathbf{n} = (n_1, n_2)$, $\mathbf{m} = (m_1, m_2) \in \mathcal{I}_2^{(n)}$ by

$$\langle \mathbf{e}_1^{\widehat{\otimes} m_1} \widehat{\otimes} \mathbf{e}_2^{\widehat{\otimes} m_2}, \mathbf{e}_1^{\widehat{\otimes} n_1} \widehat{\otimes} \mathbf{e}_2^{\widehat{\otimes} n_2} \rangle_n := (\mathbf{e}_1^{\widehat{\otimes} m_1} \widehat{\otimes} \mathbf{e}_2^{\widehat{\otimes} m_2}, \Omega_n \mathbf{e}_1^{\widehat{\otimes} n_1} \widehat{\otimes} \mathbf{e}_2^{\widehat{\otimes} n_2})_0, \tag{5.4}$$

and it defines the Hilbert space $\mathcal{H}_n := ((\mathbb{C}^2)^{\widehat{\otimes} n}, \langle \cdot, \cdot \rangle_n)$. The creation operators A_i^+ , $i = 1, 2$, are canonically defined as

$$A_1^+(\mathbf{e}_1^{\widehat{\otimes} n_1} \widehat{\otimes} \mathbf{e}_2^{\widehat{\otimes} n_2}) = \mathbf{e}_1^{\widehat{\otimes} (n_1+1)} \widehat{\otimes} \mathbf{e}_2^{\widehat{\otimes} n_2}, \quad A_2^+(\mathbf{e}_1^{\widehat{\otimes} n_1} \widehat{\otimes} \mathbf{e}_2^{\widehat{\otimes} n_2}) = \mathbf{e}_1^{\widehat{\otimes} n_1} \widehat{\otimes} \mathbf{e}_2^{\widehat{\otimes} (n_2+1)}. \tag{5.5}$$

Then it is easy to check that the annihilation operators, A_i^- , $i = 1, 2$, which are the adjoints of A_i^+ , $i = 1, 2$, respectively, are defined as

$$A_1^-(\mathbf{e}_1^{\widehat{\otimes} n_1} \widehat{\otimes} \mathbf{e}_2^{\widehat{\otimes} n_2}) = \frac{d_{n_1}^{(n)}}{d_{n_1-1}^{(n-1)}} \mathbf{e}_1^{\widehat{\otimes} (n_1-1)} \widehat{\otimes} \mathbf{e}_2^{\widehat{\otimes} n_2}, \quad \mathbf{n} = (n_1, n_2) \in \mathcal{I}_2^{(n)}, \quad n_1 \geq 1, \tag{5.6}$$

$$A_1^-(\mathbf{e}_2^{\widehat{\otimes} n}) = 0, \tag{5.7}$$

$$A_2^-(\mathbf{e}_1^{\widehat{\otimes} n_1} \widehat{\otimes} \mathbf{e}_2^{\widehat{\otimes} n_2}) = \frac{d_{n_2}^{(n)}}{d_{n_2-1}^{(n-1)}} \mathbf{e}_1^{\widehat{\otimes} n_1} \widehat{\otimes} \mathbf{e}_2^{\widehat{\otimes} (n_2-1)}, \quad \mathbf{n} = (n_1, n_2) \in \mathcal{I}_2^{(n)}, \quad n_2 \geq 1, \tag{5.8}$$

$$A_2^-(\mathbf{e}_1^{\widehat{\otimes} n}) = 0. \tag{5.9}$$

Now the commutation relations (3.14) and (3.15) are trivially satisfied. In order that the commutation relation (3.16) is satisfied, from (5.5) to (5.9), the matrix components $d_k^{(n)}$ should satisfy

$$\frac{d_{n_1}^{(n)}}{d_{n_1}^{(n-1)}} = \frac{d_{n_1+1}^{(n+1)}}{d_{n_1+1}^{(n)}}, \quad \frac{d_{n_1}^{(n+1)}}{d_{n_1-1}^{(n)}} = \frac{d_{n_1}^{(n)}}{d_{n_1-1}^{(n-1)}}, \quad n \geq 1, \quad 1 \leq n_1 \leq n-1. \tag{5.10}$$

Let us define

$$d_{n_1}^{(n)} := \prod_{k=1}^{n_1} \omega_k \prod_{l=1}^{n-n_1} \eta_l, \quad 1 \leq n_1 \leq n-1, \tag{5.11}$$

where $\{\omega_n\}_{n \geq 1}$ and $\{\eta_n\}_{n \geq 1}$ are any sequences of positive numbers. Then one checks easily that the conditions (5.10) are satisfied. We see that the measure μ which is reconstructed from $\{\Omega_n\}$ is the product measure $\mu_1 \otimes \mu_2$ with Jacobi sequences $\{\omega_n\}_{n \geq 1}$ and $\{\eta_n\}_{n \geq 1}$, respectively. See subsection 4.1.

6. Examples

6.1. Uniform measure on the unit circle

Let μ be a uniform measure on the unit circle C of xy -plane. We start by finding a system of orthonormal polynomials for μ . For each $n \geq 0$, let $u_n(x, y)$ and $v_n(x, y)$ be the real- and imaginary-parts of $(x + iy)^n$, respectively:

$$(x + iy)^n = u_n(x, y) + iv_n(x, y).$$

Notice that $u_n(x, y)$ and $v_n(x, y)$ are polynomials of x and y of degree n . For $n \geq 1$, we let $p_n(x, y) := \sqrt{2}u_n(x, y)$ and $q_n(x, y) := \sqrt{2}v_n(x, y)$.

Lemma 6.1. $\{1, p_n, q_n\}_{n=1}^\infty$ is an orthonormal system w.r.t. μ .

Proof. By denoting $z = x + iy$, we have for $m, n \geq 0$

$$\begin{aligned} \int (x + iy)^m (x + iy)^{-n} d\mu &= \frac{1}{2\pi i} \oint_C z^{m-n-1} dz \\ &= \delta_{m,n}. \end{aligned}$$

On the other hand, since $(x + iy)^m = u_m(x, y) + iv_m(x, y)$ and $(x + iy)^{-n} = (x - iy)^n = u_n(x, y) - iv_n(x, y)$ on the circle, the above integral is equal to

$$\int (u_m u_n + v_m v_n) d\mu + i \int (u_m v_n - v_m u_n) d\mu.$$

Thus we have

$$\int (u_m u_n + v_m v_n) d\mu = \delta_{m,n} \text{ and } \int (u_m v_n - v_m u_n) d\mu = 0. \tag{6.1}$$

Similarly we have the relation

$$\begin{aligned} \int (x + iy)^m (x + iy)^n d\mu &= \delta_{m+n,0} \\ &= \int (u_m u_n - v_m v_n) d\mu + i \int (u_m v_n + v_m u_n) d\mu. \end{aligned}$$

Therefore,

$$\int (u_m u_n - v_m v_n) d\mu = \delta_{m+n,0} \text{ and } \int (u_m v_n + v_m u_n) d\mu = 0. \quad (6.2)$$

The result now easily follows from (6.1) and (6.2). \square

Recall the gradation spaces $\mathcal{P}_n = \mathcal{P}_n \oplus \mathcal{P}_{n-1}$. Since we are working on two dimension, the (algebraic) dimension of \mathcal{P}_n is $n + 1$.

Lemma 6.2. *For each n , the gradation \mathcal{P}_n has an (algebraic) basis $\{p_n, q_n, (x^2 + y^2 - 1)x^\alpha y^\beta\}$, where α and β runs over $\alpha + \beta = n - 2$. Moreover, among them the vectors $(x^2 + y^2 - 1)x^\alpha y^\beta$ are zero-norm vectors.*

Proof. All the polynomials in \mathcal{P}_n are of degree n . The vectors $(x^2 + y^2 - 1)x^\alpha y^\beta$ are linearly independent and obviously of zero-norm vectors. Now by Lemma 6.1 the result follows. \square

Lemma 6.3. *For $n \geq 1$, the relations hold.*

$$\begin{aligned} xu_n &= \frac{1}{2}(u_{n+1} + u_{n-1}), & yu_n &= \frac{1}{2}(v_{n+1} - v_{n-1}) \\ xv_n &= \frac{1}{2}(v_{n+1} + v_{n-1}), & yv_n &= \frac{1}{2}(-u_{n+1} + u_{n-1}). \end{aligned}$$

Proof. From the decomposition $(x + iy)^n = u_n + iv_n$ we have the relations.

$$\begin{aligned} (u_{n+1} + iv_{n+1}) &= (x + iy)(u_n + iv_n) \\ &= (xu_n - yv_n) + i(xv_n + yu_n), \\ (u_{n-1} + iv_{n-1}) &= (x - iy)(u_n + iv_n) \\ &= (xu_n + yv_n) + i(xv_n - yu_n). \end{aligned}$$

Equating the real- and imaginary-parts in the above relations, we easily get the results. \square

We are now in a position to compute the Jacobi operators. We denote the creation operators by a_x^+ and a_y^+ and similarly for the annihilation and preservation operators. Notice that $[a_x^+, a_y^+] = 0$. We denote the constant function 1 by Φ .

Lemma 6.4. *For the orthonormal polynomials p_n and q_n in the gradation \mathcal{P}_n we have the relation.*

$$\begin{aligned} p_n(x, y) &= p_n(a_x^+, a_y^+) \Phi, \\ q_n(x, y) &= q_n(a_x^+, a_y^+) \Phi. \end{aligned}$$

Proof. From the commutativity of a_x^+ and a_y^+ we have the operator expansion:

$$(a_x^+ + ia_y^+)^n = u_n(a_x^+, a_y^+) + iv_n(a_x^+, a_y^+).$$

Then it is enough to show that

$$(x + iy)^n = (a_x^+ + ia_y^+)^n \Phi. \quad (6.3)$$

In order to prove (6.3) we use induction. Since $(a_x^+ + ia_y^+) \Phi = x + iy$, we are done with $n = 1$. Assume the relation (6.3) holds for n . Then by Lemma 6.3 we see that (6.3) holds also for $n + 1$. \square

Let $\{\mathbf{e}_1, \mathbf{e}_2\}$ be the canonical basis of \mathbb{C}^2 . For each $n \geq 0$, let $\mathcal{B}^{(n)} := \{\mathbf{b}_i^{(n)} : i = 1, \dots, n + 1\}$ be the canonical basis of $(\mathbb{C}^2)^{\widehat{\otimes} n}$ consisting of $\mathbf{e}_1^{\widehat{\otimes} i_1} \widehat{\otimes} \mathbf{e}_2^{\widehat{\otimes} i_2}$, $i_1 + i_2 = n$. Notice that the vectors of $\mathcal{B}^{(n)}$ are orthogonal to each other, but they are not normalized in general. Recall that we denote the inner product and the induced norm of $(\mathbb{C}^2)^{\widehat{\otimes} n}$ by $(\cdot, \cdot)_0$ and $|\cdot|_0$, respectively. Let $\Omega_n := [\omega_{ij}^{(n)}]_{1 \leq i, j \leq n}$ be the matrix representation of Ω_n w.r.t. $\mathcal{B}^{(n)}$. Then we have

$$\begin{aligned} \omega_{ij}^{(n)} &= \frac{1}{|\mathbf{b}_i^{(n)}|_0 |\mathbf{b}_j^{(n)}|_0} (\mathbf{b}_i^{(n)}, \Omega_n \mathbf{b}_j^{(n)})_0 \\ &= \frac{1}{|\mathbf{b}_i^{(n)}|_0 |\mathbf{b}_j^{(n)}|_0} \langle U_n \mathbf{b}_i^{(n)}, U_n \mathbf{b}_j^{(n)} \rangle_\mu. \end{aligned} \tag{6.4}$$

Let us compute the matrices of Ω_1 and Ω_2 , and Ω_3 . We have $\mathcal{B}^{(1)} = \{\mathbf{e}_1, \mathbf{e}_2\}$. Notice that $U_1 \mathbf{e}_1 = x = p_1(x, y)/\sqrt{2}$ and $U_1 \mathbf{e}_2 = y = q_1(x, y)/\sqrt{2}$. By the formula (6.4) we easily get

$$\Omega_1 = \frac{1}{2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{6.5}$$

We see that the eigenvalue of Ω_1 is $1/2$. To compute Ω_2 , let

$$\mathcal{B}^{(2)} = \{\mathbf{b}_1^{(2)}, \mathbf{b}_2^{(2)}, \mathbf{b}_3^{(2)}\} = \{\mathbf{e}_1 \widehat{\otimes} \mathbf{e}_1, \mathbf{e}_1 \widehat{\otimes} \mathbf{e}_2, \mathbf{e}_2 \widehat{\otimes} \mathbf{e}_2\},$$

in that order. We compute, for example, $\omega_{22}^{(2)}$. We have $|\mathbf{b}_2^{(2)}|_0^2 = 1/2$. By Lemma 6.3 we have

$$U_2 \mathbf{b}_2^{(2)} = \frac{1}{2\sqrt{2}} q_2(x, y).$$

Thus, by (6.4) we have $\omega_{22}^{(2)} = 1/4$. In this way we can compute all the components of Ω_2 and the result is

$$\Omega_2 = \frac{1}{8} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 1 \end{bmatrix}. \tag{6.6}$$

The eigenvalues of Ω_2 are $\{1/4, 0\}$. Next we let

$$\mathcal{B}^{(3)} = \{\mathbf{b}_1^{(3)}, \mathbf{b}_2^{(3)}, \mathbf{b}_3^{(3)}, \mathbf{b}_4^{(3)}\} = \{\mathbf{e}_1 \widehat{\otimes} \mathbf{e}_1 \widehat{\otimes} \mathbf{e}_1, \mathbf{e}_1 \widehat{\otimes} \mathbf{e}_1 \widehat{\otimes} \mathbf{e}_2, \mathbf{e}_1 \widehat{\otimes} \mathbf{e}_2 \widehat{\otimes} \mathbf{e}_2, \mathbf{e}_2 \widehat{\otimes} \mathbf{e}_2 \widehat{\otimes} \mathbf{e}_2\},$$

in that order. We notice that $|\mathbf{b}_1^{(3)}|_0^2 = |\mathbf{b}_4^{(3)}|_0^2 = 1$ and $|\mathbf{b}_2^{(3)}|_0^2 = |\mathbf{b}_3^{(3)}|_0^2 = 1/3$. By Lemma 6.3 we get

$$\begin{aligned} U_3 \mathbf{b}_1^{(3)} &= \frac{1}{4\sqrt{2}} p_3(x, y), & U_3 \mathbf{b}_2^{(3)} &= \frac{1}{4\sqrt{2}} q_3(x, y), \\ U_3 \mathbf{b}_3^{(3)} &= -\frac{1}{4\sqrt{2}} p_3(x, y), & U_3 \mathbf{b}_4^{(3)} &= -\frac{1}{4\sqrt{2}} q_3(x, y). \end{aligned}$$

Thus by (6.4) we get

$$\Omega_3 = \frac{1}{32} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 3 & 0 & -3 \\ -3 & 0 & 3 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}. \tag{6.7}$$

We can compute that the eigenvalues of Ω_3 are $\{1/8, 0\}$.

It turns out that the rank of Ω_n is 2. Thus, we don't need to have such a big matrix of size $n + 1$ for the representation of Ω_n . Below we find a reduced form of Ω_n .

Lemma 6.5. *The isomorphism operator $U_n : (\mathbb{C}^d)^{\widehat{\otimes} n} \rightarrow \mathcal{P}_n$ is given by*

$$U_n(\widehat{\mathbf{e}}_1^{i_1} \widehat{\otimes} \dots \widehat{\otimes} \widehat{\mathbf{e}}_d^{i_d}) = x_1^{i_1} \dots x_d^{i_d} - P_{n-1}(x_1^{i_1} \dots x_d^{i_d}), \quad i_1 + \dots + i_d = n.$$

Proof. We use induction. For $n = 1$,

$$U_1(\mathbf{e}_i) = a_i^+ \Phi = P_1 x_i = x_i - \langle 1, x_i \rangle = x_i - P_0(x_i).$$

Suppose the statement of the Lemma holds for n . Without loss it is enough to check the relation for $\widehat{\mathbf{e}}_1^{i_1+1} \widehat{\otimes} \widehat{\mathbf{e}}_2^{i_2} \dots \widehat{\otimes} \widehat{\mathbf{e}}_d^{i_d}$, $i_1 + \dots + i_d = n$. By the induction assumption,

$$\begin{aligned} & U_{n+1}(\widehat{\mathbf{e}}_1^{i_1+1} \widehat{\otimes} \widehat{\mathbf{e}}_2^{i_2} \dots \widehat{\otimes} \widehat{\mathbf{e}}_d^{i_d}) \\ &= a_{x_1}^+ (a_{x_1}^+)^{i_1} \dots (a_{x_d}^+)^{i_d} \Phi \\ &= a_{x_1}^+ U_n(\widehat{\mathbf{e}}_1^{i_1} \widehat{\otimes} \dots \widehat{\otimes} \widehat{\mathbf{e}}_d^{i_d}) \\ &= P_{n+1}(x_1^{i_1+1} \dots x_d^{i_d} - x_1 P_{n-1}(x_1^{i_1} \dots x_d^{i_d})) \\ &= x_1^{i_1+1} \dots x_d^{i_d} - P_n(x_1^{i_1+1} \dots x_d^{i_d}) - P_{n+1}(x_1 P_{n-1}(x_1^{i_1} \dots x_d^{i_d})) \\ &= x_1^{i_1+1} \dots x_d^{i_d} - P_n(x_1^{i_1+1} \dots x_d^{i_d}), \end{aligned}$$

where we have used $P_{n+1}(x_1 P_{n-1}(x_1^{i_1} \dots x_d^{i_d})) = 0$ because $x_1 P_{n-1}(x_1^{i_1} \dots x_d^{i_d}) \in P_n$ and \mathcal{P}_{n+1} is orthogonal to \mathcal{P}_n . \square

Recall from Lemma 6.2 that the basis of \mathcal{P}_n consists of vectors p_n , q_n , and r_n 's, where r_n is any function of the form $(x^2 + y^2 - 1)x^\alpha y^\beta$ with $\alpha + \beta = n - 2$. We let $p_n^{\widehat{\otimes}}(\mathbf{e}_1, \mathbf{e}_2)$, $q_n^{\widehat{\otimes}}(\mathbf{e}_1, \mathbf{e}_2)$, and $r_n^{\widehat{\otimes}}(\mathbf{e}_1, \mathbf{e}_2)$ be the unique elements of $(\mathbb{C}^2)^{\widehat{\otimes} n}$ such that their image under U_n are $p_n(x, y)$, $q_n(x, y)$, and $r_n(x, y)$, respectively. By using Lemma 6.5 it is obvious to see how they look like. Indeed, it inherits the form only from the part of degree- n monomials. For example, for $p_3(x, y) = \sqrt{2}(x^3 - 3xy^2)$, since $P_2(\sqrt{2}(x^3 - 3xy^2)) = 0$, $p_3^{\widehat{\otimes}}(\mathbf{e}_1, \mathbf{e}_2) = \sqrt{2}(\mathbf{e}_1 \widehat{\otimes} \mathbf{e}_1 \widehat{\otimes} \mathbf{e}_1 - 3\mathbf{e}_1 \widehat{\otimes} \mathbf{e}_2 \widehat{\otimes} \mathbf{e}_2)$. For $r_n(x, y) = (x^2 + y^2 - 1)x^\alpha y^\beta$, $\alpha + \beta = n - 2$, since $r_n(x, y) = (x^2 + y^2)x^\alpha y^\beta - P_{n-1}((x^2 + y^2)x^\alpha y^\beta)$, we have $r_n^{\widehat{\otimes}}(\mathbf{e}_1, \mathbf{e}_2) = (\mathbf{e}_1 \widehat{\otimes} \mathbf{e}_1 + \mathbf{e}_2 \widehat{\otimes} \mathbf{e}_2) \widehat{\otimes} \widehat{\mathbf{e}}_1^\alpha \widehat{\otimes} \widehat{\mathbf{e}}_2^\beta$. Notice that the part of degree- n monomials of $r_n(x, y)$ is $(x^2 + y^2)x^\alpha y^\beta$ and from this the form of $r_n^{\widehat{\otimes}}(\mathbf{e}_1, \mathbf{e}_2)$ inherits.

Definition 6.6. We say that a linearly independent set $\mathcal{C}^{(n)} = \{\mathbf{c}_1, \dots, \mathbf{c}_k\}$, $1 \leq k \leq n + 1$, is closed for Ω_n if for any vector $\mathbf{c}_i \in \mathcal{C}^{(n)}$, $\Omega_n \mathbf{c}_i$ is a linear combination of the vectors of $\mathcal{C}^{(n)}$.

Once one has any closed independent subset $\mathcal{C}^{(n)}$ for Ω_n , then it is enough to represent Ω_n in the basis of $\mathcal{C}^{(n)}$.

Proposition 6.7. *For each $n \geq 0$, $\mathcal{C}^{(n)} := \{p_n^{\widehat{\otimes}}(\mathbf{e}_1, \mathbf{e}_2), q_n^{\widehat{\otimes}}(\mathbf{e}_1, \mathbf{e}_2)\}$ is closed for Ω_n and $\widetilde{\Omega}_n$, the representation of Ω_n in the basis of $\mathcal{C}^{(n)}$, is given by*

$$\widetilde{\Omega}_n = \begin{bmatrix} \frac{1}{|p_n^{\widehat{\otimes}}(\mathbf{e}_1, \mathbf{e}_2)|^2} & 0 \\ 0 & \frac{1}{|q_n^{\widehat{\otimes}}(\mathbf{e}_1, \mathbf{e}_2)|^2} \end{bmatrix}. \tag{6.8}$$

Proof. It is easy to see that $p_n^{\widehat{\otimes}}(\mathbf{e}_1, \mathbf{e}_2)$, $q_n^{\widehat{\otimes}}(\mathbf{e}_1, \mathbf{e}_2)$, and $r_n^{\widehat{\otimes}}(\mathbf{e}_1, \mathbf{e}_2)$'s constitute the orthogonal basis of $(\mathbb{C}^2)^{\widehat{\otimes}n}$. By definition we see that

$$U_n(p_n^{\widehat{\otimes}}(\mathbf{e}_1, \mathbf{e}_2)) = p_n(x, y),$$

and similar relations for q_n and r_n 's. For any $r_n^{\widehat{\otimes}}(\mathbf{e}_1, \mathbf{e}_2)$, we have

$$(p_n^{\widehat{\otimes}}(\mathbf{e}_1, \mathbf{e}_2), \Omega_n r_n^{\widehat{\otimes}}(\mathbf{e}_1, \mathbf{e}_2))_0 = \langle U_n(p_n^{\widehat{\otimes}}(\mathbf{e}_1, \mathbf{e}_2)), U_n(r_n^{\widehat{\otimes}}(\mathbf{e}_1, \mathbf{e}_2)) \rangle_\mu = \langle p_n, r_n \rangle_\mu = 0.$$

Similarly we have $(q_n^{\widehat{\otimes}}(\mathbf{e}_1, \mathbf{e}_2), \Omega_n r_n^{\widehat{\otimes}}(\mathbf{e}_1, \mathbf{e}_2))_0 = 0$. Thus we see that $\mathcal{C}^{(n)}$ is closed for Ω_n . The representation (6.8) follows directly from the definition of Ω_n . \square

Here are some examples. We have $\mathcal{C}^{(1)} = \{\sqrt{2}\mathbf{e}_1, \sqrt{2}\mathbf{e}_2\}$ and $|\sqrt{2}\mathbf{e}_1|_0^2 = |\sqrt{2}\mathbf{e}_2|_0^2 = 2$. Thus

$$\tilde{\Omega}_1 = \frac{1}{2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{6.9}$$

We have $\mathcal{C}^{(2)} = \{\sqrt{2}(\mathbf{e}_1 \widehat{\otimes} \mathbf{e}_1 - \mathbf{e}_2 \widehat{\otimes} \mathbf{e}_2), 2\sqrt{2}(\mathbf{e}_1 \widehat{\otimes} \mathbf{e}_2)\}$ and

$$|\sqrt{2}(\mathbf{e}_1 \widehat{\otimes} \mathbf{e}_1 - \mathbf{e}_2 \widehat{\otimes} \mathbf{e}_2)|_0^2 = 4 = |2\sqrt{2}(\mathbf{e}_1 \widehat{\otimes} \mathbf{e}_2)|_0^2.$$

Thus,

$$\tilde{\Omega}_2 = \frac{1}{4} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{6.10}$$

Now for $n = 3$, we have $\mathcal{C}^{(3)} = \{\sqrt{2}(\mathbf{e}_1 \widehat{\otimes} \mathbf{e}_1 \widehat{\otimes} \mathbf{e}_1 - 3\mathbf{e}_1 \widehat{\otimes} \mathbf{e}_2 \widehat{\otimes} \mathbf{e}_2), \sqrt{2}(3\mathbf{e}_1 \widehat{\otimes} \mathbf{e}_1 \widehat{\otimes} \mathbf{e}_2 - \mathbf{e}_2 \widehat{\otimes} \mathbf{e}_2 \widehat{\otimes} \mathbf{e}_2)\}$ and

$$|\sqrt{2}(\mathbf{e}_1 \widehat{\otimes} \mathbf{e}_1 \widehat{\otimes} \mathbf{e}_1 - 3\mathbf{e}_1 \widehat{\otimes} \mathbf{e}_2 \widehat{\otimes} \mathbf{e}_2)|_0^2 = 8 = |\sqrt{2}(3\mathbf{e}_1 \widehat{\otimes} \mathbf{e}_1 \widehat{\otimes} \mathbf{e}_2 - \mathbf{e}_2 \widehat{\otimes} \mathbf{e}_2 \widehat{\otimes} \mathbf{e}_2)|_0^2.$$

Thus,

$$\tilde{\Omega}_3 = \frac{1}{8} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{6.11}$$

We remark that the non-zero spectrum of Ω_n 's and $\tilde{\Omega}_n$'s are equal to each other for $n = 1, 2, 3$. Of course it must be the case for any n .

6.2. Uniform measure on the half circle

Let μ be the probability measure uniformly distributed on the half circle on the xy -plane; $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1, y \geq 0\}$. Let us find orthogonal polynomials for this measure. Let $u_n(x, y)$ and $v_n(x, y)$ be the polynomials introduced in the previous subsection, i.e., they satisfy the equation $(x + iy)^n = u_n(x, y) + iv_n(x, y)$. It turns out that the gradation structure for this measure is very similar to that of the uniform measure on the circle, which we investigated in the previous section. For each $n \geq 1$ let us define the following polynomials.

$$r_n(x, y) := \begin{cases} u_n(x, y), & n, \text{ odd} \\ u_{n/2}(x^2 - y^2, 2xy), & n, \text{ even} \end{cases},$$

$$s_n(x, y) := \begin{cases} v_n(x, y), & n, \text{ odd} \\ v_{n/2}(x^2 - y^2, 2xy), & n, \text{ even} \end{cases}.$$

For each $n \geq 1$, let $\mathcal{Q}_n := \{r_n, s_n\}$. We have the following result.

Lemma 6.8. *For any $n, m \geq 1$, $\mathcal{Q}_n \perp \mathcal{Q}_m$ if both n and m are even, or both of them are odd.*

Proof. We deal separately with odd and even cases. First observe from the definition that

$$\begin{aligned} (x - iy)^n &= u_n(x, -y) + iv_n(x, -y) \\ &= u_n(x, y) - iv_n(x, y). \end{aligned}$$

From this we get

$$u_n(x, -y) = u_n(x, y), \quad v_n(x, -y) = -v_n(x, y). \quad (6.12)$$

Similarly we get

$$u_n(-x, y) = (-1)^n u_n(x, y), \quad v_n(-x, y) = (-1)^{n+1} v_n(x, y). \quad (6.13)$$

Therefore, if n is odd, we get

$$u_n(-x, y) = -u_n(x, y), \quad v_n(-x, y) = v_n(x, y), \quad (n, \text{ odd}). \quad (6.14)$$

Let us just show the orthogonality of r_n and s_m . When n and m are odd, by (6.12) and (6.14), we have $u_n(-x, -y) = -u_n(x, y)$ and $v_m(-x, -y) = -v_m(x, y)$. That is, the product $u_n v_m$ is symmetric w.r.t. the origin and hence when we integrate out the product $u_n v_m$ over the unit circle, the integral on the upper half circle and the integral on the lower half circle are the same. Thus,

$$\begin{aligned} 0 &= \frac{1}{2\pi} \int_0^{2\pi} u_n(\cos \theta, \sin \theta) v_m(\cos \theta, \sin \theta) d\theta \\ &= \frac{1}{\pi} \int_0^\pi u_n(\cos \theta, \sin \theta) v_m(\cos \theta, \sin \theta) d\theta \\ &= \int u_n(x, y) v_m(x, y) d\mu(x, y). \end{aligned}$$

Now for each $n, m \geq 1$, we see by change of variables that

$$\begin{aligned} &\int u_n(x^2 - y^2, 2xy) v_m(x^2 - y^2, 2xy) d\mu(x, y) \\ &= \frac{1}{\pi} \int_0^\pi u_n(\cos 2\theta, \sin 2\theta) v_m(\cos 2\theta, \sin 2\theta) d\theta \end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{2\pi} \int_0^{2\pi} u_n(\cos \theta, \sin \theta) v_m(\cos \theta, \sin \theta) d\theta \\
 &= 0.
 \end{aligned}$$

So, if n and m are both even, then $\langle r_n, s_m \rangle_\mu = 0$. This ends the proof. \square

Now we can state gradation spaces for the measure μ .

Proposition 6.9. *For $n \geq 1$, the gradation space \mathcal{P}_n has a (algebraic) basis $\{p_n, q_n, (x^2 + y^2 - 1)x^\alpha y^\beta : \alpha + \beta = n - 2\}$, where $p_n := r_n - P_{n-1}r_n$ and $q_n := s_n - P_{n-1}s_n$.*

Proof. Note that any vector of the form $(x^2 + y^2 - 1)x^\alpha y^\beta$, $\alpha + \beta = n - 2$, is a polynomial of degree n and it is a μ -zero norm vector. p_n and q_n are monomials of degree n , and altogether they have full rank for \mathcal{P}_n . We complete the proof by Lemma 6.8. \square

From Proposition 6.9 the following holds.

Corollary 6.10. *For the uniform measure on the half circle, the ranks of Ω_n are all 2 for $n \geq 1$.*

6.3. Moments of uniform measure on the unit circle

In this subsection we revisit the example of uniform measure on the unit circle which we discussed in subsection 6.1. Here we compute the CAP operators and find a formula for the moments. We have seen that the gradation spaces \mathcal{P}_n has dimension 2 consisting of orthonormal basis $\{p_n, q_n\}$ for $n \geq 1$. Notice that once the creation operators come from the measure, that is by the relation $A_i^+ = U^* a_i^+ U$, it is easy to see that $\|A_i^+|_{\mathcal{H}_n}(\xi_n)\|_{n+1} = 0$ whenever $\|\xi_n\|_n = 0$. Thus it is enough and very convenient if we represent $A_i^+|_{\mathcal{H}_n} : \mathcal{H}_n \rightarrow \mathcal{H}_{n+1}$ w.r.t. an orthonormal basis, whenever we can find it easily. Recall the notations

$$p_n^{\widehat{\otimes}}(\mathbf{e}_1, \mathbf{e}_2) := U_n^*(p_n), \quad q_n^{\widehat{\otimes}}(\mathbf{e}_1, \mathbf{e}_2) := U_n^*(q_n).$$

Then $\mathcal{B}_n := \{p_n^{\widehat{\otimes}}(\mathbf{e}_1, \mathbf{e}_2), q_n^{\widehat{\otimes}}(\mathbf{e}_1, \mathbf{e}_2)\}$ constitutes an orthonormal basis for \mathcal{H}_n . We have

Lemma 6.11. *By using the bases \mathcal{B}_n ($\mathcal{B}_0 := \{1\}$) above we have the following matrix representation for CAP operators.*

$$\begin{aligned}
 A_1^+|_{\mathcal{H}_n} &= \frac{1}{2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad n \geq 1, \quad A_1^+|_{\mathcal{H}_0} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \\
 A_2^+|_{\mathcal{H}_n} &= \frac{1}{2} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad n \geq 1, \quad A_2^+|_{\mathcal{H}_0} = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \\
 A_1^-|_{\mathcal{H}_n} &= \frac{1}{2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad n \geq 2, \quad A_1^-|_{\mathcal{H}_1} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 \end{bmatrix}, \\
 A_2^-|_{\mathcal{H}_n} &= \frac{1}{2} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad n \geq 2, \quad A_2^-|_{\mathcal{H}_1} = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 1 \end{bmatrix}, \\
 A_1^0 &= 0, \quad A_2^0 = 0.
 \end{aligned}$$

Proof. The proof follows easily from Lemma 6.3. \square

As an example let us compute $\int x^2y^2d\mu(x, y)$:

$$\begin{aligned} \int x^2y^2d\mu(x, y) &= \langle 1, x^2y^21 \rangle_\mu \\ &= \langle \Phi_0, X_1^2X_2^2\Phi_0 \rangle_0 \\ &= \langle \Phi_0, (A_1^+ + A_1^-)^2(A_2^+ + A_2^-)^2\Phi_0 \rangle_0 \\ &= \langle \Phi_0, (A_1^-A_1^+A_2^-A_2^+ + A_1^-A_1^-A_2^+A_2^+)\Phi_0 \rangle_0 \\ &= \frac{1}{4} - \frac{1}{8} = \frac{1}{8}. \end{aligned}$$

In the last line we have used the formula in Lemma 6.11 and in the line before it, we notice that among all 16 terms there are only two terms that contribute to the integral. By directly computing, we get also $\int x^2y^2d\mu(x, y) = 1/8$.

7. Marginals

In this section we discuss the marginals of a given measure. Let μ be a probability measure on \mathbb{R}^d . For any $1 \leq k < d$, let $\mathcal{S} = \{i_1, \dots, i_k\} \subset \{1, \dots, d\}$ be a subset. Without loss we may assume $\mathcal{S} = \{1, \dots, k\}$. Let $\mu^{(\mathcal{S})}$ be the marginal of μ onto $\prod_{i \in \mathcal{S}} \mathbb{R}$. That is, for any Borel set $A \subset \prod_{i \in \mathcal{S}} \mathbb{R}$, $\mu^{(\mathcal{S})}(A) := \mu(A \times (\prod_{i \notin \mathcal{S}} \mathbb{R}))$. From the general theory developed in sections 3-5, it is straightforward how to construct the CAP operators and form generators (operators Ω_n in (5.1)) for $\mu^{(\mathcal{S})}$. Let $\mathcal{P}^{(\mathcal{S})}$ be the space of all polynomials of x_i for $i = 1, \dots, k$. Likely we let $\mathcal{P}_n^{(\mathcal{S})}$ be the space of all polynomials of $x_i, i = 1, \dots, k$, of degree less than or equal to n . $\mathcal{P}_n^{(\mathcal{S})}$ denotes the n th gradation space:

$$\mathcal{P}_n^{(\mathcal{S})} := \mathcal{P}_n^{(\mathcal{S})} \ominus \mathcal{P}_{n-1}^{(\mathcal{S})}.$$

As before we let $P_n^{(\mathcal{S})}$ and $P_n^{(\mathcal{S})}$ the projections onto $\mathcal{P}_n^{(\mathcal{S})}$ and $\mathcal{P}_n^{(\mathcal{S})}$, respectively. We notice that $\mathcal{P}_n^{(\mathcal{S})}$ is a subspace of \mathcal{P}_n and for any $(n_1, \dots, n_k) \in \mathcal{I}_k^{(n)}$, the vector $(a_1^+)^{n_1} \dots (a_k^+)^{n_k} \Phi = x_1^{n_1} \dots x_k^{n_k} - P_{n-1}(x_1^{n_1} \dots x_k^{n_k})$ belongs to \mathcal{P}_n , but it may not equal to $(a_1^{+, \mathcal{S}})^{n_1} \dots (a_k^{+, \mathcal{S}})^{n_k} \Phi = x_1^{n_1} \dots x_k^{n_k} - P_{n-1}^{(\mathcal{S})}(x_1^{n_1} \dots x_k^{n_k})$, where $a_i^{+, \mathcal{S}}$ s are creation operators for $\mu_{\mathcal{S}}$. Now we define CAP operators by

$$\begin{aligned} a_i^{+, \mathcal{S}} \Big|_{\mathcal{P}_n^{(\mathcal{S})}} &:= \mathcal{P}_{n+1}^{(\mathcal{S})} x_i \mathcal{P}_n^{(\mathcal{S})}, \\ a_i^{-, \mathcal{S}} \Big|_{\mathcal{P}_n^{(\mathcal{S})}} &:= \left(a_i^{+, \mathcal{S}} \Big|_{\mathcal{P}_{n-1}^{(\mathcal{S})}} \right)^*, \\ a_i^{0, \mathcal{S}} \Big|_{\mathcal{P}_n^{(\mathcal{S})}} &:= x_i - a_i^{+, \mathcal{S}} \Big|_{\mathcal{P}_n^{(\mathcal{S})}} - a_i^{-, \mathcal{S}} \Big|_{\mathcal{P}_n^{(\mathcal{S})}}, \end{aligned}$$

for $i = 1, \dots, k$. These operators enable us to define the form generator $\Omega_n^{(\mathcal{S})}$: for $(n_1, \dots, n_k) \in \mathcal{I}_k^{(n)}$ and $(m_1, \dots, m_k) \in \mathcal{I}_k^{(n)}$,

$$\begin{aligned} &\left(\widehat{e}_1^{\otimes n_1} \widehat{\otimes} \dots \widehat{\otimes} e_k^{\otimes n_k}, \Omega_n^{(\mathcal{S})} e_1^{\otimes m_1} \widehat{\otimes} \dots \widehat{\otimes} e_k^{\otimes m_k} \right)_0 \\ &:= \left\langle (a_1^{+, \mathcal{S}})^{n_1} \dots (a_k^{+, \mathcal{S}})^{n_k} \Phi, (a_1^{+, \mathcal{S}})^{m_1} \dots (a_k^{+, \mathcal{S}})^{m_k} \Phi \right\rangle_\mu. \end{aligned} \tag{7.1}$$

In the right hand side, the integration w.r.t. μ is equal to the integration w.r.t. $\mu^{(\mathcal{S})}$ because the integrand is a function of variables x_i for $i \in \mathcal{S}$. Below we consider some examples.

Product measures. Let $\mu := \mu_1 \otimes \mu_2$ on \mathbb{R}^2 where μ_1 and μ_2 are one-dimensional measures with Jacobi sequences $(\{\omega_n\}, \{\alpha_n\})$ and $(\{\eta_n\}, \{\beta_n\})$, respectively. Let $\mathcal{S} := \{1\} \subset \{1, 2\}$. Then obviously $\mu^{(\mathcal{S})} = \mu_1$. We will recover this by constructing $\Omega_n^{(\mathcal{S})}$ in (7.1). Let $\{p_n(x_1)\}$ be the orthogonal polynomials for μ_1 satisfying the three-term recurrence relation in (2.1). By using the fact that Ω_n 's are diagonal, as noted in Example 5.2, we can inductively see that

$$x_1^n - P_{n-1}^{(\mathcal{S})}(x_1^n) = p_n(x_1).$$

Therefore, $\omega_n^{(\mathcal{S})}$, the matrix component of 1×1 matrix $\Omega_n^{(\mathcal{S})}$, is equal to $\langle p_n(x_1), p_n(x_1) \rangle_\mu = \prod_{k=1}^n \omega_k$. This is the Jacobi coefficients of μ_1 .

Uniform measure on the unit circle. We come back to the uniform measure on the unit circle discussed in subsection 6.1. Let μ be the uniform measure on the unit circle and let $\mathcal{S} := \{1\} \subset \{1, 2\}$. We want to compute $\mu^{(\mathcal{S})}$. Recall the notations in subsection 6.1:

$$(x + iy)^n = u_n(x, y) + iv_n(x, y),$$

and $p_n(x, y) = \sqrt{2}u_n(x, y)$, $q_n(x, y) = \sqrt{2}v_n(x, y)$, which are orthonormal functions for μ . The following lemma will be useful.

Lemma 7.1. *On the unit circle $x^2 + y^2 = 1$, $u_n(x, y)$ is a polynomial of x , say $\tilde{u}_n(x)$, of degree n and the coefficient of the leading term is 2^{n-1} .*

By directly computing a few number of functions we see that

$$\begin{aligned} u_1(x, y) &= \tilde{u}_1(x) = x, \\ u_2(x, y) &= \tilde{u}_2(x) = 2x^2 - 1, \\ u_3(x, y) &= \tilde{u}_3(x) = 4x^3 - 3x, \\ u_4(x, y) &= \tilde{u}_4(x) = 8x^4 - 8x^2 + 1, \end{aligned}$$

and so on.

Proof of Lemma 7.1. On the unit circle, using polar coordinates we get

$$u_n(x, y) = \text{Re}(e^{in\theta}) = \cos n\theta.$$

Recall an identity for trigonometric functions:

$$\cos(n + 1)\theta = 2 \cos n\theta \cos \theta - \cos(n - 1)\theta.$$

The statement of the lemma is shown by an induction with the above identity. \square

Proposition 7.2. *For $n \geq 1$ let $p_n^{(\mathcal{S})}(x) := 2^{-(n-1)}\tilde{u}_n(x)$ and let $p_0^{(\mathcal{S})}(x) := 1$. Then $\{p_n^{(\mathcal{S})}(x) : n \geq 0\}$ is orthogonal polynomials for $\mu^{(\mathcal{S})}$. Moreover, the Jacobi coefficients of $\mu^{(\mathcal{S})}$ are $\{\omega_n\}_{n=1}^\infty = \{1/2, 1/4, 1/4, \dots\}$ and $\alpha_n = 0$. Therefore $\mu^{(\mathcal{S})}$ is the Kesten distribution $\mu_{1/2, 1/4}$, or an arcsine law with density $\frac{1}{\pi} \frac{1}{\sqrt{1-x^2}}$, $|x| < 1$.*

Proof. $u_n(x, y)$ belongs to \mathcal{P}_n , the n th gradation space for the original measure μ . Now by Lemma 7.1, $u_n(x, y) = \tilde{u}_n(x)$ is also a polynomial of the variable x only, thus $\tilde{u}_n(n)$ belongs to $\mathcal{P}_n^{(\mathcal{S})}$, that is $p_n^{(\mathcal{S})}(x) :=$

$2^{-(n-1)}\tilde{u}_n(x) = x^n - P_{n-1}^{(S)}(x^n)$, that is $p_n^{(S)}(x) = \left(a_1^{+,S}\right)^n \Phi$. In order to compute the Jacobi coefficients, we see that for $n \geq 1$

$$\begin{aligned} \prod_{k=1}^n \omega_k &= \left\langle \left(a_1^{+,S}\right)^n \Phi, \left(a_1^{+,S}\right)^n \Phi \right\rangle_{\mu_D} \\ &= \left\langle p_n^{(S)}(x)\Phi, p_n^{(S)}(x)\Phi \right\rangle_{\mu_D} \\ &= 2^{-2(n-1)} \langle u_n(x, y), u_n(x, y) \rangle_{\mu} \\ &= 2^{-2(n-1)} \frac{1}{2}. \end{aligned}$$

Thus we get $\{\omega_n\}_{n=1}^\infty = \{1/2, 1/4, 1/4, \dots\}$. \square

A non-symmetric measure. Let $\mu := \frac{1}{4}(\delta_{(2,0)} + \delta_{(1,1)} + \delta_{(0,0)} + \delta_{(1,-1)})$, a point mass on \mathbb{R}^2 . We notice that μ is the rotation of the product measure $\nu_1 \otimes \nu_2$, where $\nu_1 = \nu_2 = \frac{1}{2}(\delta_{1/\sqrt{2}} + \delta_{-1/\sqrt{2}})$, followed by a translation by 1 in the x -axis. Since the orthonormal system of $\nu_1 \otimes \nu_2$ is $\{1, \sqrt{2}, \sqrt{2}, 2xy\}$, the orthonormal system of μ is $\{1, (x-1) - y, (x-1) + y, (x-1)^2 - y^2\}$. Since we aim at the x -marginal, we may rewrite the orthogonal system in the following way. For the degree 1 polynomials we use linear combinations and for the degree 2 polynomial we use the identity $(x-1)^2 + y^2 = 1$ which holds for μ -a.e.. Thus we have another orthonormal system for μ of the form

$$\{1, \sqrt{2}(x-1), \sqrt{2}y, 2(x-1)^2 - 1\}.$$

Thus, the orthonormal polynomials for $\mu^{(1)}$, the x -marginal of μ , are

$$\{1, \sqrt{2}(x-1), 2(x-1)^2 - 1\}.$$

Thus the monic bases [1] for μ and $\mu^{(1)}$ are

$$\left\{1, (x-1), y, (x-1)^2 - \frac{1}{2}\right\} \text{ and } \left\{1, (x-1), (x-1)^2 - \frac{1}{2}\right\}, \tag{7.2}$$

respectively. From (7.2) we easily compute the Jacobi operators as follows.

$$\Omega_0 = 1, \quad \Omega_1 = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}, \quad \Omega_2 = \begin{bmatrix} \frac{1}{4} & 0 & -\frac{1}{4} \\ 0 & 0 & 0 \\ -\frac{1}{4} & 0 & \frac{1}{4} \end{bmatrix}$$

and

$$\Omega_0^{(1)} = 1, \quad \Omega_1^{(1)} = \frac{1}{2}, \quad \Omega_2^{(1)} = \frac{1}{4}.$$

We promptly see that $\mu^{(1)} = \frac{1}{4}\delta_0 + \frac{1}{2}\delta_1 + \frac{1}{4}\delta_2$ and the Jacobi sequences are $\{\omega_n\} = \{1/2, 1/2, 0, \dots\}$ and $\{\alpha_n\} = \{1, 1, 1, 0, \dots\}$.

8. Deficiency rank of Jacobi operator and support of the measure

In the examples of section 6, we see that the rank of Ω_n is 2 for all $n \geq 1$, i.e., it is uniformly bounded by a constant, or at least, it is less than d_n , the possible full rank of Ω_n . Below we discuss this phenomenon.

On \mathbb{R}^d , we say that a subset $S \subset \mathbb{R}^d$ is an algebraic level surface if there is a polynomial p such that $S = \{(x_1, \dots, x_d) \in \mathbb{R}^d : p(x_1, \dots, x_d) = 0\}$.

Definition 8.1. Let μ be a probability measure on \mathbb{R}^d with finite moments of all orders. By defining $\rho_n := \text{rank } \Omega_n$, we call $\rho := (\rho_n)_{n \geq 1}$ the rank sequence of μ . We say that μ has deficiency rank if there is n_0 such that $\text{rank } \Omega_{n_0}$ is strictly less than $d_{n_0} = \binom{n_0+d-1}{d-1}$, the possible maximum rank of Ω_{n_0} .

Notice that once $\text{rank } \Omega_{n_0} < d_{n_0}$, it is the case for all $n \geq n_0$.

Theorem 8.2. Let μ be a probability measure on \mathbb{R}^d with finite moments of all orders. Then μ has deficiency rank if and only if the measure μ is supported on an algebraic level surface.

Proof. Suppose that μ has deficiency rank. Then Ω_n has an eigenvalue 0 with corresponding eigenvector, say $\xi \in (\mathbb{C}^d)^{\otimes n}$. Let $p := U_n(\xi) \in \mathcal{P}_n$. Then,

$$\begin{aligned} \int |p|^2 d\mu &= \langle p, p \rangle_\mu \\ &= \langle \xi, \xi \rangle_n \\ &= (\xi, \Omega_n \xi)_0 \\ &= 0. \end{aligned} \tag{8.1}$$

This means that $p = 0$ μ -a.e. Therefore μ is supported on the algebraic level surface $\{p = 0\}$. Conversely, suppose that μ is supported on an algebraic level surface, say $\{p = 0\}$, for a polynomial p of degree n . We may assume $p \in \mathcal{P}_n$. Let $\xi = U^*(p) \in \mathcal{H}_n$. By the equality (8.1) we have

$$0 = \int |p|^2 d\mu = (\xi, \Omega_n \xi)_0.$$

Thus $\sqrt{\Omega_n} \xi = 0$, and hence $\Omega_n \xi = \sqrt{\Omega_n}^2 \xi = 0$, i.e., Ω_n has a zero eigenvalue and therefore $\text{rank } \Omega_n < d_n$. This ends the proof. \square

Remark 8.3. (1) When μ has deficiency rank, the ranks of Ω_n may be uniformly bounded by a constant or may increase monotonically. The case of uniform measure on the unit circle is an example of uniform bound. Now consider the measure $d\mu(x, y)d\pi(z)$ on \mathbb{R}^3 . Here $d\mu(x, y)$ is the uniform measure on the unit circle on the xy -plane and $d\pi(z)$ is a measure of infinite orthogonal polynomials, e.g., a Gaussian measure on the z -axis. Let $\{p_n(x, y), q_n(x, y)\}_{n \geq 0}$ be the orthogonal systems for $\mu(dx dy)$ as above and let $\{r_n(z)\}_{n \geq 0}$ be the orthogonal polynomials for $\pi(dz)$. Then the orthogonal system for \mathcal{P}_n is $\{p_k(x, y)r_{n-k}(z)\}_{k=0}^n \cup \{q_k(x, y)r_{n-k}(z)\}_{k=0}^n$. So, the rank of Ω_n is $2(2n + 1)$ which is less than $d_n = \binom{n+2}{2} = (n + 2)(n + 1)/2$, and hence the measure $d\mu(x, y)d\pi(z)$ has deficiency rank, but the ranks increase to infinity. Notice that the cylinder $\{x^2 + y^2 = 1\}$ is an algebraic level surface on \mathbb{R}^3 .

(2) Although the ranks may increase to infinity for a measure of deficiency rank, the increase is negligible in the sense that $\lim_{n \rightarrow \infty} \text{rank } \Omega_n / d_n = 0$. In fact, suppose that μ is supported on an algebraic level surface $p = 0$, where p is a polynomial of degree k . Then the dimension of null space of \mathcal{P}_n is at least $\binom{n-k+d-1}{d-1}$ for $n \geq k$. Thus $\lim_{n \rightarrow \infty} \text{rank } \Omega_n / d_n = 0$ since $\lim_{n \rightarrow \infty} \binom{n-k+d-1}{d-1} / d_n = 1$.

Example 8.4. (1) Any measure with finitely many point masses has deficiency rank because there always exists an algebraic level surface that contains all the mass points.

(2) Let μ_1 be a discrete measure with support the natural numbers, e.g., let $\mu_1 = \sum_{n=1}^\infty \frac{1}{2^n} \delta_n$. Let μ_2 be a copy of μ_1 and let $d\mu(x, y) := d\mu_1(x)d\mu_2(y)$ be the product measure. We see that the support of μ is the

two-dimensional lattice points on the first quadrant. Now there is no algebraic level curve that contains all the lattice points on the first quadrant. In fact, let $\{p_n(x)\}_{n \geq 0}$ be the orthogonal polynomials for μ_1 . Then an orthogonal system for \mathcal{P}_n for μ is $\{p_k(x)p_{n-k}(y)\}_{k=0}^n$, which is of dimension $n + 1 = d_n$. Thus μ is not a measure of deficiency rank.

(3) Let $d\mu(x, y)$ be the image measure on the curve $y = \sin x$ of the Gaussian measure on the x -axis by the map $x \mapsto (x, \sin x)$. Notice that μ is singular w.r.t. Lebesgue measure on the plane (it lives on a curve). But μ is not a measure of deficiency rank, because any polynomial of x and y can't be a zero function on the curve $y = \sin x$.

Let us now more closely look at the relation between the deficiency rank and support of the measure. As we have seen in Theorem 8.2, if a measure μ has deficiency rank then there exist polynomials with zero norm. Let μ be a probability measure on \mathbb{R}^d which has finite moments of all orders. Suppose that μ has deficiency rank and let

$$\mathcal{N} \equiv \mathcal{N}_\mu := \{p \in \mathcal{P} : \int |p|^2 d\mu = 0\}.$$

We call $p \in \mathcal{N}$ a *base null polynomial* (simply base) if p has no more factor of zero norm, i.e., there is no pair $h \in \mathcal{N}$ and $p_1 \in \mathcal{P}$ such that degree of p_1 is greater than or equal to 1 and $p = hp_1$. When p is a base, the level surface $\{p = 0\}$ we call a *base level surface*.

We recall some of basic facts for polynomial algebra.

Proposition 8.5. *Any ideal of the polynomial algebra \mathcal{P} is finitely generated.*

The above fundamental result on polynomial rings traces back to Hilbert. In fact, the coefficients of our polynomials are taken from the real number field which is Noetherian, so is the polynomial.

We also recall the following

Proposition 8.6. *The polynomial algebra \mathcal{P} is a unique factorization ring.*

Hence for the null kernel \mathcal{N} there exist a finite number of polynomials f_1, \dots, f_k such that

$$\mathcal{N} = \sum_{i=1}^k f_i \mathcal{P}, \tag{8.2}$$

where f_1, \dots, f_k are linearly independent. The algebraic set corresponding to \mathcal{N} is defined by

$$S(\mathcal{N}) = \bigcap_{i=1}^k \{f_i = 0\}, \tag{8.3}$$

where

$$\{f_i = 0\} = \{(x_1, x_2, \dots, x_d) \in \mathbb{R}^d : f_i(x_1, x_2, \dots, x_d) = 0\}.$$

The support of μ is a closed subset of \mathbb{R}^d defined by

$$\text{supp } \mu = \mathbb{R}^d \setminus \bigcup \{U : \text{an open set in } \mathbb{R}^d \text{ such that } \mu(U) = 0\}$$

The following is a fundamental relation between deficiency rank and support of the measure.

Theorem 8.7. Let μ be a probability measure on \mathbb{R}^d with finite moments of all orders. If μ has deficiency rank then $\text{supp } \mu \subset S(\mathcal{N})$.

Proof. It follows easily from Theorem 8.2. \square

Example 8.8. (1) Let μ be the uniform measure on the unit circle or the uniform measure on the half circle of \mathbb{R}^2 . We have seen that in both cases the polynomial $x^2 + y^2 - 1$ is the unique base null polynomial. Thus in two cases the measures are supported on this base level surface; $x^2 + y^2 = 1$.

(2) On \mathbb{R}^2 , let $\mu = \frac{1}{4}(\delta_{(1,1)} + \delta_{(-1,1)} + \delta_{(-1,-1)} + \delta_{(1,-1)})$. It is not hard to see that the base null polynomials are $x^2 - 1$, $y^2 - 1$, $x^2 - y^2$, and $x^2 + y^2 - 2$. (the last two are the linear combinations of the first two). Now the intersection of the base level surfaces are exactly four points $\{(1, 1), (-1, 1), (-1, -1), (1, -1)\}$, the support of the measure. In general, let $\{\mathbf{a}_i = (a_1^i, \dots, a_d^i) \in \mathbb{R}^d : i = 1, \dots, k\}$ be a finite subset and let $\mu := \sum_{i=1}^k m_i \delta_{\mathbf{a}_i}$ with $\sum_{i=1}^k m_i = 1$. Then for each $j = 1, \dots, d$, the polynomial $\prod_{i=1}^k (x_j - a_j^i)$ is a base null polynomial. The intersection of the base level surfaces $\prod_{i=1}^k (x_j - a_j^i) = 0$, $j = 1, \dots, d$, gives rise to the support of the measure.

Acknowledgments

A. Dhahri acknowledges support by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education (NRF-2016R1C1B1010008). The research by H.J. Yoo was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education (NRF-2016R1D1A1B03936006).

References

- [1] L. Accardi, A. Barhoumi, A. Dhahri, Identification of the theory of orthogonal polynomials in d -indeterminates with the theory of 3-diagonal symmetric interacting Fock spaces on \mathbb{C}^d , *Infin. Dimens. Anal. Quantum Probab. Relat. Top.* 20 (1) (2017) 1750004.
- [2] L. Accardi, M. Bożejko, Interacting Fock space and Gaussianization of probability measures, *Infin. Dimens. Anal. Quantum Probab. Relat. Top.* 1 (1998) 663–670.
- [3] L. Accardi, H.-H. Kuo, A.I. Stan, Characterization of probability measures through canonically associated interacting Fock spaces, *Infin. Dimens. Anal. Quantum Probab. Relat. Top.* 7 (4) (2004) 485–505.
- [4] L. Accardi, M. Nhani, Interacting Fock spaces and orthogonal polynomials in several variables, in: N. Obata, A. Hora, T. Matsui (Eds.), *The Crossroad of Non-commutativity, Infinite-Dimensionality*, World Scientific, 2002.
- [5] T. Chihara, *An Introduction to Orthogonal Polynomials*, Gordon and Breach, New York, 1978.
- [6] C.F. Dunkl, Y. Xu, *Orthogonal Polynomials of Several Variables*, Cambridge University Press, 2001.
- [7] A. Hora, N. Obata, *Quantum Probability and Spectral Analysis of Graphs*, Springer, Berlin, 2007.
- [8] P.E.T. Jorgensen, Essential self-adjointness of semibounded operators, *Math. Ann.* 237 (1978) 187–192.
- [9] M.A. Kowalski, The recurrence formulas for polynomials in n variables, *SIAM J. Math. Anal.* 13 (1982) 309–315.
- [10] M.A. Kowalski, Orthogonality and recursion formulas for polynomials in n variables, *SIAM J. Math. Anal.* 13 (1982) 316–323.
- [11] H.L. Krall, I.M. Sheffer, Orthogonal polynomials in two variables, *Ann. Pure Appl. Math.* 76 (4) (1967) 325–376.
- [12] E. Nelson, Analytic vectors, *Ann. Math.* 70 (1978) 572–615.
- [13] M. Reed, B. Simon, *Methods of Modern Mathematical Physics, I. Functional Analysis*, Academic Press, Inc., 1980.
- [14] P.K. Suetin, *Orthogonal Polynomials in Two Variables*, Gordon and Breach Science Publishers, Amsterdam, 1999, xx+348.
- [15] Y. Xu, Unbounded commutative operators and multivariate orthogonal polynomials, *Proc. Amer. Math. Soc.* 119 (4) (1993) 1223–1231.
- [16] Y. Xu, Multivariate orthogonal polynomials and operator theory, *Trans. Amer. Math. Soc.* 343 (1) (1994) 193–202.
- [17] Y. Xu, Block Jacobi matrices and zeros of multivariate orthogonal polynomials, *Trans. Amer. Math. Soc.* 342 (2) (1994) 855–866.
- [18] Y. Xu, Recurrence formulas for multivariate orthogonal polynomials, *Math. Comp.* 62 (206) (1994) 687–702.
- [19] Y. Xu, Lecture notes on orthogonal polynomials of several variables, in: W. zu Castell, F. Filbir, B. Forster (Eds.), *Inzell Lectures on Orthogonal Polynomials*, in: *Advances in the Theory of Special Functions and Orthogonal Polynomials*, vol. 2, Nova Science Publishers, 2004.
- [20] G. Szegő, *Orthogonal Polynomials*, 4th ed., Amer. Math. Soc., Providence, RI, 1975.